Proof of Ore's Theorem*

Here is a more carefully explained proof of Ore's Theorem than the one given in lectures. The first two steps are illustrated by the attached example. *This proof may be considered non-examinable.*

Theorem 3.9 (Ore). *Let G be a simple graph on n vertices. If $n \geq 3$, and*

$$\delta(x) + \delta(y) \geq n$$

for each pair of non-adjacent vertices x and y, then G has a closed Hamiltonian path.

Proof. Suppose, for a contradiction, that G does not have a closed Hamiltonian path.

1. Pick any any two vertices of G which aren't already joined by an edge, and add a new edge between them. Keep on doing this until we reach a graph G_{last} which *does* have a closed Hamiltonian path. (The process must stop because eventually we will reach the complete graph on n vertices, which obviously has a closed Hamiltonian path.)

2. Let $\bar{G}$ be the graph obtained immediately before G_{last}, and suppose that $\{x, y\}$ is the edge added to $\bar{G}$ to obtain G_{last}.

Let $(z_1, \ldots, z_n, z_1)$ be a closed Hamiltonian path in G_{last}. This must use the edge $\{x, y\}$ at some point (otherwise $\bar{G}$ would have a closed Hamiltonian path, and there would have been no need to consider G_{last}). If $\{z_n, z_1\} = \{x, y\}$ then $(z_1, \ldots, z_n)$ is a non-closed Hamiltonian path in $\bar{G}$. Otherwise there is some r such that $1 \leq r < n$ and $z_r = x$ and $z_{r+1} = y$; now

$$(z_{r+1}, \ldots, z_n, z_1, \ldots, z_r)$$

is a non-closed Hamiltonian path in $\bar{G}$. Note that either way, all the edges used in this path appear in $\bar{G}$: it is only $\{x, y\}$ that appears in G_{last} but not in $\bar{G}$. Relabel the vertices so that this path is $(x_1, \ldots, x_n)$.

3. Suppose we could find a vertex x_i such that x is adjacent to x_i, and y is adjacent to x_{i-1}. Then

$$(x, x_i, \ldots, x_{n-1}, y, x_{i-1}, \ldots, x)$$

would be a closed Hamiltonian path in $\bar{G}$, a contradiction.

Aside: It is at this point that we need $n \geq 3$: if $n = 2$ then the first step is (x, y), and the second is (y, x), which means we have used an edge twice. Paths are, in particular, trails, so they aren't allowed to repeat edges. As long as $n \geq 3$ this problem doesn't arise.

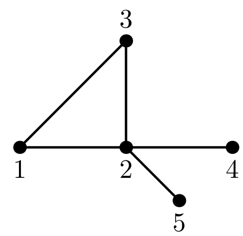
4. It remains to show that there must be such a vertex x_i. This is where we need the hypothesis on degrees. Since $\bar{G}$ is obtained from G by adding edges, it still satisfies this hypothesis. Let

$$A = \{i : 2 \le i \le n \text{ and } x_i \text{ is adjacent to } x\},$$
$$B = \{i : 2 \le i \le n \text{ and } x_{i-1} \text{ is adjacent to } y\}.$$

As our graphs have no loops, $|A| = \delta(x)$ and $|B| = \delta(y)$. As x and y are not adjacent in $\bar{G}$ (recall that $\{x, y\}$ was added to $\bar{G}$ to obtain G_{last}), our hypothesis tells us that $\delta(x) + \delta(y) \ge n$.

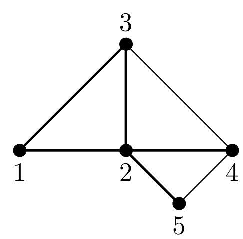
Hence A and B are subsets of $\{2, \ldots, n\}$ containing at least n elements between them. It follows that they must intersect non-trivially. If $i \in A \cap B$ then x_i is a suitable vertex for step 3. □

Example: Let $n = 5$. The graph below has vertex set $\{1, 2, 3, 4, 5\}$ and edges $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{2, 4\}$, $\{2, 5\}$.



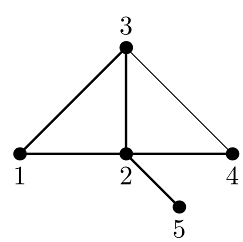
(This graph doesn't satisfy the hypothesis on the degrees, but we don't use this until step 4. This saves drawing a large number of edges which would be irrelevant in steps 1 and 2.)

1. We might first add the edge $\{3, 4\}$. The resulting graph still doesn't have a closed Hamiltonian path, so we add another edge, say $\{4, 5\}$. This gives the graph



which has $(1, 2, 5, 4, 3, 1)$ as a closed Hamiltonian path. (So $z_1 = 1$, $z_2 = 2$, $z_3 = 5$, $z_4 = 4$, $z_5 = 3$.)

2. The last edge added is $\{x, y\} = \{4, 5\}$ so $\bar{G}$ is as shown below.



Starting with the closed path $(1, 2, 5, 4, 3, 1)$ in G_{final} we find that $r = 3$, $x = 5$, $y = 4$. The resulting non-closed Hamiltonian path in $\bar{G}$ is $(4, 3, 1, 2, 5)$. So in the relabelling step we take $x_1 = 4$, $x_2 = 3$, $x_3 = 1$, $x_4 = 2$, $x_5 = 5$.