A LOWER BOUND FOR THE PARTITION FUNCTION FROM CHEBYSHEV'S INEQUALITY APPLIED TO A COIN FLIPPING MODEL FOR THE RANDOM PARTITION

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ABSTRACT. We use a coin flipping model for the random partition and Chebyshev's inequality to prove the lower bound $\lim \frac{\log p(n)}{\sqrt{n}} \geq C$ for the number of partitions $p(n)$ of n, where C is an explicit constant.

A *partition* of *size* $n \in \mathbf{N}_0$ is a decreasing sequence of natural numbers whose sum is n. Let $p(n)$ be the number of partitions of n. For example, $p(5) = 7$ counts the partitions (5), $(4,1)$, $(3,2)$, $(3,1,1)$, $(2,2,1)$, $(2,1,1,1)$ and $(1,1,1,1,1)$. In this note we use a model for the random partition to prove that for all $\varepsilon > 0$

$$(\star) \qquad \frac{\log p(n)}{\sqrt{n}} > \frac{\sqrt{8}\log 2}{1+\varepsilon} \quad \text{for all } n \text{ sufficiently large.}$$

We end with an explicit bound that replaces $\sqrt{8}\log 2$ with the slightly smaller constant $\frac{8}{3}\log 2$. The proof of $(\star)$ is self-contained and intended to be readable by anyone knowing the basics of probability theory.

The asymptotically correct result is $\lim_{n\to\infty} \frac{\log p(n)}{\sqrt{n}} = 2\sqrt{\pi^2/6}$. The upper bound $\log p(n) \leq 2\sqrt{\pi^2/6}\sqrt{n}$ is relatively easy to prove—see for instance Theorem 15.7 in [5]—but getting a tight lower bound is much more challenging. A fairly lengthy proof using only real analysis was given by Erdős in [2]. Our proof is motivated by the model for the random partition in [1, §4.3], and by the abacus notation for partitions (see [3, page 79]). The latter was used in [4] to prove the uniform lower bound $p(n) \geq \mathrm{e}^{2\sqrt{n}}/14$, and in [6] to prove the upper bound $\log p(n) \leq C(\varepsilon)n^{\frac{1}{2}+\varepsilon}$ for all $\varepsilon > 0$. The novel feature here is to combine these motivations to give a simple proof of $(\star)$.

The proof begins with a coin flipping model for the random partition. Using linearity of expectation it is easy to show that a partition generated by m flips has expected size about $m^2/8$. Critically, the standard deviation is of order $m^{3/2}$. By Chebyshev's inequality, most of the 2^m partitions generated by m coin flips have size within a few standard deviations of $m^2/8$. This leads quickly to the claimed bound.

Coin flipping model. We represent a partition λ of length ℓ as the set of *boxes* $\{(i,j) : 1 \leq i \leq \ell, 1 \leq j \leq \lambda_i\}$, forming its *Young diagram*. We draw

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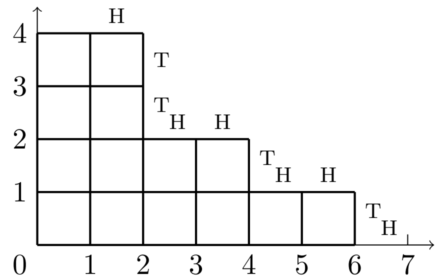

FIGURE 1. The Young diagram of the partition $(6, 4, 2, 2)$ with the corresponding coin flip sequence HTTHHTHHTH.

Young diagrams in 'French notation', so that the box (i, j) is geometrically a unit square with diagonal from $(i-1, j-1)$ to (i, j). For example, the partition $(6, 4, 2, 2)$ of length 4 is shown in Figure 1 above.

Let $\Omega = \{\mathrm{H}, \mathrm{T}\}^m$ be the probability space for m flips of an unbiased coin in which each $\omega \in \Omega$ has equal probability $\frac{1}{2^m}$. Given $\omega \in \Omega$ with exactly ℓ tails, we define the boundary of a corresponding partition $P(\omega)$ of length ℓ as follows. Start at $(0, \ell)$ and step right to $(1, \ell)$. Then for each head, step one unit right, and for each tail, step one unit down. For instance if $m = 10$ and $\omega = \mathrm{HTTHHTHHTH}$ then $P(\omega) = (6, 4, 2, 2)$; the final head corresponds to a step from $(6, 0)$ to $(7, 0)$ that is not part of a geometric box.

Let N be the size of $P(\omega)$ and let X_t be the number of heads up to and including flip t. Let $Y = m - X_m$ be the total number of tails; this is the length of $P(\omega)$. A move down at step t adds $X_{t-1} + 1$ boxes to the Young diagram. Therefore setting

$$C_t = \begin{cases} X_{t-1} & \text{if } \omega_t = \mathrm{T} \\ 0 & \text{if } \omega_t = \mathrm{H} \end{cases}$$

we have $N = Y + \sum_{t=1}^{m} C_t$.

Expectation and variance. Since X_t is distributed binomially as $\mathrm{Bin}(t, \frac{1}{2})$, we have $\mathbf{E}[X_t] = t/2$ and $\mathbf{Var}\, X_t = t/4$. Hence $\mathbf{E}[Y] = m/2$ and $\mathbf{Var}\, Y = m/4$. Observe that C_t is non-zero only if flip t is tails. Conditioning on this event shows that $\mathbf{E}[C_t] = \frac{1}{2}\mathbf{E}[X_{t-1}] = \frac{t-1}{4}$. Hence, by linearity of expectation, $\mathbf{E}[N] = \mathbf{E}[Y] + \sum_{t=1}^{m} \mathbf{E}[C_t] = \frac{m}{2} + \frac{1}{4}\sum_{t=1}^{m}(t-1) = m/2 + m(m-1)/8 = m(m+3)/8$.

Lemma 1. *If $t \leq u$ then the random variables C_t and X_u are uncorrelated.*

Proof. Again we condition on the event that flip t is tails. In this event, $C_t = X_{t-1}$ and $X_u = X_{t-1} + W$, where W is the number of heads between flips $t+1$ and u, inclusive. Since W is independent of X_{t-1},

$$\mathbf{E}[C_t X_u] = \frac{1}{2}\mathbf{E}[X_{t-1}(X_{t-1} + W)]$$

$$\begin{aligned}
&= \frac{1}{2}\big(\mathbf{E}[X_{t-1}^2] + \mathbf{E}[X_{t-1}]\mathbf{E}[W]\big) \\
&= \frac{1}{2}\big(\mathbf{Var}\, X_{t-1} + \mathbf{E}[X_{t-1}]^2 + \mathbf{E}[X_{t-1}]\mathbf{E}[W]\big) \\
&= \frac{1}{2}\left(\frac{t-1}{4} + \Big(\frac{t-1}{2}\Big)^2 + \Big(\frac{t-1}{2}\Big)\Big(\frac{u-t}{2}\Big)\right) \\
&= \frac{1}{2}\Big(\frac{t-1}{4}\Big)\big(1 + t - 1 + u - t\big) \\
&= \mathbf{E}[C_{t-1}]\mathbf{E}[X_u]
\end{aligned}$$

as required. □

As a corollary, we find that

$$\mathbf{E}[C_t C_u] = \frac{1}{2}\mathbf{E}[C_t X_{u-1}] = \frac{1}{2}\mathbf{E}[C_t]\mathbf{E}[X_{u-1}] = \mathbf{E}[C_t][C_u]$$

whenever $t < u$. Hence C_t and C_u are uncorrelated for distinct t and u. This is perhaps a little surprising, since the inequality $C_t \le C_u$, which holds whenever $t < u$ and $C_u \ne 0$, shows that they are not in general independent. A final conditioning argument shows that $\mathbf{Var}\, C_t = \mathbf{E}[C_t^2] - \mathbf{E}[C_t]^2 = \frac{1}{2}\mathbf{E}[X_{t-1}^2] - \frac{1}{4}\mathbf{E}[X_{t-1}]^2$, and so

$$\begin{aligned}
\mathbf{Var}\, C_t = \frac{1}{2}\mathbf{Var}\, X_{t-1} + \frac{1}{4}\mathbf{E}[X_{t-1}]^2 &= \frac{1}{2}\Big(\frac{t-1}{4}\Big) + \frac{1}{4}\Big(\frac{t-1}{2}\Big)^2 \\
&= \Big(\frac{t-1}{16}\Big)\big(2 + t - 1\big) = \frac{t^2-1}{16}.
\end{aligned}$$

By Lemma 1, $\mathbf{E}[C_t Y] = \mathbf{E}[C_t(m - X_m)] = \mathbf{E}[C_t]\mathbf{E}[m - X_m] = \mathbf{E}[C_t]\mathbf{E}[Y]$ for all t. Hence C_t and Y are also uncorrelated. If Z and Z' are uncorrelated random variables then, by a one-line calculation, $\mathbf{Var}(Z + Z') = \mathbf{Var}\, Z + \mathbf{Var}\, Z'$. We therefore have $\mathbf{Var}\, N = \mathbf{Var}\, Y + \sum_{t=1}^m \mathbf{Var}\, C_t$ and so

$$\mathbf{Var}\, N = \frac{m}{4} + \sum_{t=1}^{m} \frac{t^2-1}{16} = \frac{3m}{16} + \frac{m(m+1)(2m+1)}{96} = \frac{m^3}{48} + \frac{m^2}{32} + \frac{19m}{96}.$$

Critically $\mathbf{Var}\, N$ is cubic in m, not quartic as one might naively expect. To simplify calculations, we use the upper bound $m^3/48 + m^2/32 + 19m/96 \le 2m^3/96 + 4m^3/96 = m^3/16$ for $m \ge 3$ to get $\mathbf{Var}\, N \le m^3/16$.

Lower bound. The concentration of measure estimate in Chebyshev's inequality

$$\mathbf{P}\Big[\,\big|Z - \mathbf{E}[Z]\big| \ge d\sqrt{\mathbf{Var}\, Z}\,\Big] \le \frac{1}{d^2}$$

implies that

$$\mathbf{P}\Big[\big|N - \frac{m(m+3)}{8}\big| \ge d\frac{m^{3/2}}{4}\Big] \le \frac{1}{d^2}$$

for $m \ge 3$ and any $d > 0$.

The probability space Ω has 2^m elements. The proportion giving partitions with $\left|N-m(m+3)/8\right| < dm^{3/2}/4$ is more than $1-1/d^2$. Since distinct coin flip sequences give distinct partitions, it follows that

$$\sum_n p(n) > 2^m\big(1-\frac{1}{d^2}\big)$$

where the sum is over all $n \in \mathbf{N}_0$ such that $|n - m(m+3)/8| < dm^{3/2}/4$. Since $p(n)$ is monotonic, we deduce that, for $m \geq 3$,

$$p\Big(\frac{m(m+3)}{8} + d\frac{m^{3/2}}{4}\Big) > \frac{2^m}{dm^{3/2}}2\big(1-\frac{1}{d^2}\big),$$

where we extend the domain of p to $\mathbf{R}$ by setting $p(x) = p(\lfloor x \rfloor)$. The function $d \mapsto \frac{1}{d}\big(1-\frac{1}{d^2}\big)$ is maximized when $d=\sqrt{3}$, where it has value $\frac{2}{3\sqrt{3}}$. Therefore we take $d=\sqrt{3}$. Let $\eta > 0$ be given. Provided m is sufficiently large we have $3m/8 + \sqrt{3}m^{3/2}/4 < \eta m^2/8$. Hence

$$(\dagger) \qquad p\Big(\frac{m^2}{8}(1+\eta)\Big) > 2^m \frac{4}{3\sqrt{3}m^{3/2}}$$

for all m sufficiently large. Setting $n = m^2(1+\eta)/8$ and taking logs we obtain

$$\log p(n) \geq \sqrt{\frac{8n}{1+\eta}}\log 2 - \frac{3}{2}\log\frac{8n}{1+\eta} + \log\frac{4}{3\sqrt{3}}$$

for all n sufficiently large. Since $(\log n)/\sqrt{n} \to 0$ as $n \to \infty$ it follows that for all $\varepsilon > 0$,

$$\frac{\log p(n)}{\sqrt{n}} > \frac{\sqrt{8}\log 2}{1+\varepsilon}$$

for all n sufficiently large, as claimed in $(\star)$. The constant on the right-hand side is approximately 1.961, somewhat lower than the asymptotically correct $2\sqrt{\pi^2/6} \approx 2.565$. For a concrete lower bound, take $\eta = \frac{1}{8}$ and $m = 8\sqrt{n}/3$ in $(\dagger)$ to get $p(n) \geq 2^{8\sqrt{n}/3}/2^{5/2}n^{3/4}$ for all n sufficiently large. (One can easily check that $n \geq 10^6$ suffices.) Using a computer to check small cases one can show that in fact

$$p(n) \geq \frac{2^{8\sqrt{n}/3}}{2^{5/2}n^{3/4}} \quad \text{for all } n \geq 2.$$

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