A LOWER BOUND FOR THE PARTITION FUNCTION FROM CHEBYSHEV'S INEQUALITY APPLIED TO A COIN FLIPPING MODEL FOR THE RANDOM PARTITION

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Abstract. We use a coin flipping model for the random partition and Chebyshev's inequality to prove the lower bound $\lim_{n\to\infty} \frac{\log p(n)}{\sqrt{n}} \geq C$ for the number of partitions $p(n)$ of n, where C is an explicit constant.

A *partition* of *size* $n \in \mathbf{N}$ is a decreasing sequence of natural numbers whose sum is n. Let $p(n)$ be the number of partitions of n. For example, $p(5) = 7$ counts the partitions (5), $(4,1)$, $(3,2)$, $(3,1,1)$, $(2,2,1)$, $(2,1,1,1)$ and $(1,1,1,1,1)$ each of size 5. In this note we use a model for the random partition to prove that for each $\epsilon > 0$

$$\frac{\log p(n)}{\sqrt{n}} > \frac{\sqrt{8}\log 2}{1+\epsilon} \quad \text{for all } n \text{ sufficiently large.} \tag{1}$$

(Throughout 'log' means 'logarithm to base e'.) We end with an explicit bound in which $\sqrt{8}\log 2$ is replaced with the slightly smaller constant $\frac{8}{3}\log 2$.

The asymptotically correct result is

$$\lim_{n\to\infty} \frac{\log p(n)}{\sqrt{n}} = 2\sqrt{\frac{\pi^2}{6}}.$$

The upper bound $\log p(n) \leq 2\sqrt{\pi^2/6}\sqrt{n}$ is relatively easy to prove — see for instance Theorem 15.7 in [6] — but getting a tight lower bound is more challenging. A somewhat delicate proof using only real analysis was given by Erdős in [2]. Our proof of (1) is motivated by the model for the random partition in [1, §4.3], and by James' abacus notation for partitions (see [3, page 79]). The latter was used in [4] to prove the uniform lower bound $p(n) \geq \mathrm{e}^{2\sqrt{n}}/14$, and in [7] to prove the upper bound $\log p(n) \leq A(\epsilon)n^{\frac{1}{2}+\epsilon}$ for all $\epsilon > 0$. The novel feature here is to combine these motivations to give a simple proof of (1), intended to be readable by anyone who has taken first courses in analysis and probability. To illustrate the key ideas, we begin by giving an informal overview of a particular case of the proof.

Informal overview. Let Ω be the probability space for 80 flips of a fair coin. Below we describe a coin flipping model in which each sequence in Ω defines a different partition. Let N be the size of this partition. Now N is a random variable, so the size of the partition varies, but by (4) and (5)

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below, the expected value of N is 830 and its standard deviation is 104.32. One may reasonably expect that at least 95% of partitions defined by the coin flipping model have size within three standard deviations of the mean. In mathematical notation,

$$\mathbf{P}\big[|N - 830| < 313\big] \geq 1 - \frac{1}{20}. \tag{2}$$

The rigorous formulation of (2), see (6) in the proof below, is obtained using Chebyshev's inequality. The general theme is 'concentration of measure'. A rightly sceptical reader may be more immediately convinced by the results from the simulation shown in Figure 1 overleaf: of one million random partitions, each defined by 80 coin flips, 99.8% had size strictly between 517 and 1143.

We now make the critical step, *from probability to counting*. In this example, the probability space Ω consists of 2^{80} coin flip sequences. By (2), at least $(1-\frac{1}{20})2^{80}$ of these coin flip sequences define a partition of size strictly between 517 and 1143. Since different coin flip sequences define different partitions, we conclude that there are at least $(1-\frac{1}{20})2^{80}$ partitions whose size is in $\{518, 519, \ldots, 1142\}$. In mathematical notation,

$$\sum_{n=518}^{1142} p(n) \geq (1 - \tfrac{1}{20})2^{80}. \tag{3}$$

Since $p(n)$ is positive increasing, it follows that $\big|\{518, 519, \ldots, 1142\}\big| p(1142) \geq (1-\frac{1}{20})2^{80}$, and so $p(1142) \geq \frac{19}{20\times 625}2^{80}$. The important feature is that $p(1142)$ is of roughly the same order as 2^{80}. See (7) for the general (and rigorous) version, which leads quickly to (1).

Coin flipping model. We now begin the proof. We represent a partition λ of length ℓ as the set of *boxes* $\{(i,j) : 1 \leq i \leq \ell, 1 \leq j \leq \lambda_i\}$, forming its *Young diagram*. We draw Young diagrams in 'French notation', so that the box (i,j) is geometrically a unit square with diagonal from $(i-1, j-1)$ to (i,j). For example, the Young diagram of the partition $(6,4,2,2)$ of size 14 and length 4 is shown in Figure 2 overleaf.

Fix $m \in \mathbf{N}$ and let $\Omega = \{\mathrm{H}, \mathrm{T}\}^m$ be the probability space for m flips of an unbiased coin in which each $\omega \in \Omega$ has equal probability $\frac{1}{2^m}$. Given $\omega \in \Omega$ with exactly ℓ tails, we define the boundary of a corresponding partition $Q(\omega)$ of length ℓ as follows. Start at $(0,\ell)$ and step right to $(1,\ell)$. Then for each head, step one unit right, and for each tail, step one unit down. For instance if $m = 10$ and $\omega = \mathrm{HTTHHTHHTH}$ then $Q(\omega) = (6,4,2,2)$ as shown in Figure 2 overleaf; the final head corresponds to the step from $(6,0)$ to $(7,0)$ that is not part of a geometric box. Let $N(\omega)$ be the size of $Q(\omega)$. We emphasise that Q and N are random variables: formally Q is a function from Ω to the set of partitions, and $N : \Omega \to \mathbf{N}$ is a function from Ω to the set of natural numbers.

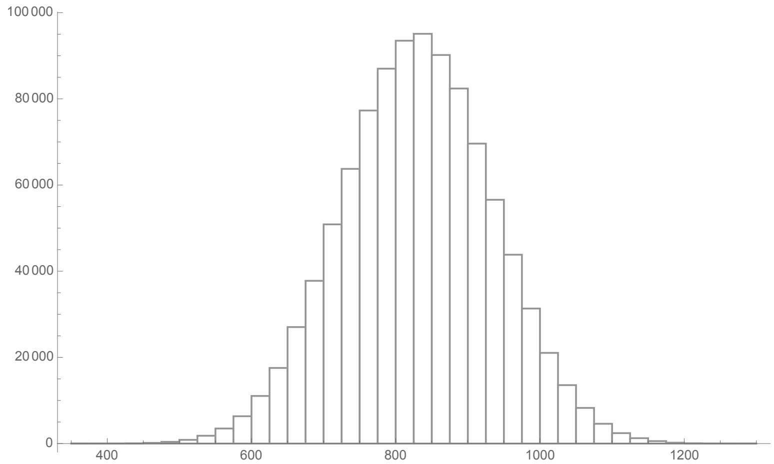

FIGURE 1. Histogram showing the sizes of 1 000 000 random partitions each defined by 80 flips in the coin flipping model. Equations (4) and (5) predict a mean size of 830 and a standard deviation of 104.320; in fact the mean is 829.967 and the standard deviation is 104.319. The least size is 359, the greatest size is 1320, and 997 659 of the partitions have size in the set $\{518, 519, \ldots, 1142\}$. The Mathematica [5] notebook used for this simulation is available from the author's website: `www.ma.rhul.ac.uk/~uvah099/`.

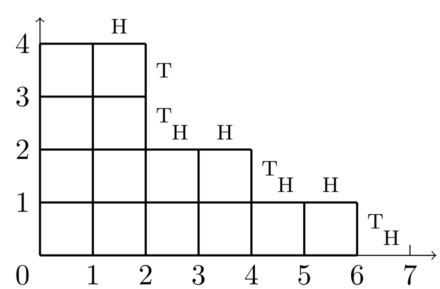

FIGURE 2. The coin flip sequence $\omega =$ HTTHHTHHTH defines the partition $Q(\omega) = (6, 4, 2, 2)$ of size $N(\omega) = 14$.

Let $X_t(\omega)$ be the number of heads up to and including flip t in a coin flip sequence ω. Let $Y(\omega) = m - X_m(\omega)$ be the total number of tails; this is the length of the partition $Q(\omega)$. A move down at step t adds $X_{t-1} + 1$ boxes to the Young diagram. Therefore defining

$$C_t(\omega) = \begin{cases} X_{t-1}(\omega) & \text{if } \omega_t = \mathrm{T} \\ 0 & \text{if } \omega_t = \mathrm{H} \end{cases}$$

we have $N = Y + \sum_{t=1}^{m} C_t$.

Expectation and variance. Since the random variable X_t is distributed binomially as $\mathrm{Bin}(t, \frac{1}{2})$, we have $\mathbf{E}[X_t] = t/2$ and $\mathbf{Var}\, X_t = t/4$. Since $Y = m - X_t$ it follows that $\mathbf{E}[Y] = m/2$ and $\mathbf{Var}\, Y = m/4$. Observe that C_t is non-zero only if flip t is tails. Conditioning on this event by considering the two possibilities for ω_t we get

$$\mathbf{E}[C_t] = \mathbf{P}[\omega_t = T]\mathbf{E}[X_{t-1}] + \mathbf{P}[\omega_t = \mathrm{H}]0 = \frac{1}{2}\frac{t-1}{2} + \frac{1}{2}0 = \frac{t-1}{4}.$$

Hence, by linearity of expectation,

(4)

$$\mathbf{E}[N] = \mathbf{E}[Y] + \sum_{t=1}^{m} \mathbf{E}[C_t] = \frac{m}{2} + \frac{1}{4}\sum_{t=1}^{m}(t-1) = \frac{m}{2} + \frac{m(m-1)}{8} = \frac{m(m+3)}{8}.$$

Lemma 1. If $t \leq u$ then the random variables C_t and X_u are uncorrelated.

Proof. Again we condition on the event that flip t is tails. In this event, $C_t = X_{t-1}$ and $X_u = X_{t-1} + W$, where W is the number of heads between flips $t+1$ and u, inclusive. Otherwise $C_t = 0$. Since W is independent of X_{t-1}, and W is distributed binomially as $\mathrm{Bin}(u-t, \frac{1}{2})$, it follows that

$$\begin{aligned}
\mathbf{E}[C_t X_u] &= \frac{1}{2}\mathbf{E}[X_{t-1}(X_{t-1} + W)] \\
&= \frac{1}{2}\big(\mathbf{E}[X_{t-1}^2] + \mathbf{E}[X_{t-1}W]\big) \\
&= \frac{1}{2}\big(\mathbf{Var}\, X_{t-1} + \mathbf{E}[X_{t-1}]^2 + \mathbf{E}[X_{t-1}]\mathbf{E}[W]\big) \\
&= \frac{1}{2}\left(\frac{t-1}{4} + \Big(\frac{t-1}{2}\Big)^2 + \Big(\frac{t-1}{2}\Big)\Big(\frac{u-t}{2}\Big)\right) \\
&= \frac{1}{2}\Big(\frac{t-1}{4}\Big)\big(1 + (t-1) + (u-t)\big) \\
&= \frac{t-1}{4}\frac{u}{2} \\
&= \mathbf{E}[C_{t-1}]\mathbf{E}[X_u]
\end{aligned}$$

as required. $\square$

As a corollary, we find using the same conditioning argument that

$$\mathbf{E}[C_t C_u] = \frac{1}{2}\mathbf{E}[C_t X_{u-1}] = \frac{1}{2}\mathbf{E}[C_t]\mathbf{E}[X_{u-1}] = \mathbf{E}[C_t]\mathbf{E}[C_u]$$

whenever $t < u$. Hence C_t and C_u are uncorrelated for distinct t and u. This is perhaps a little surprising, since the inequality $C_t \leq C_u$, which holds whenever $t < u$ and $C_u \neq 0$, shows that they are not in general independent. A final conditioning argument shows that $\mathbf{Var}\, C_t = \mathbf{E}[C_t^2] -$

$\mathbf{E}[C_t]^2 = \frac{1}{2}\mathbf{E}[X_{t-1}^2] - \frac{1}{4}\mathbf{E}[X_{t-1}]^2$, and so

$$\begin{aligned}\mathbf{Var}\, C_t = \frac{1}{2}\,\mathbf{Var}\, X_{t-1} + \frac{1}{4}\mathbf{E}[X_{t-1}]^2 &= \frac{1}{2}\Big(\frac{t-1}{4}\Big) + \frac{1}{4}\Big(\frac{t-1}{2}\Big)^2 \\ &= \Big(\frac{t-1}{16}\Big)\big(2+t-1\big) = \frac{t^2-1}{16}.\end{aligned}$$

By Lemma 1, $\mathbf{E}[C_t Y] = \mathbf{E}[C_t(m-X_m)] = \mathbf{E}[C_t]\mathbf{E}[m-X_m] = \mathbf{E}[C_t]\mathbf{E}[Y]$ for all t. Hence C_t and Y are also uncorrelated. If Z and Z' are uncorrelated random variables then, by a one-line calculation, $\mathbf{Var}(Z+Z') = \mathbf{Var}\, Z + \mathbf{Var}\, Z'$. We therefore have $\mathbf{Var}\, N = \mathbf{Var}\, Y + \sum_{t=1}^m \mathbf{Var}\, C_t$ and so

(5)

$$\mathbf{Var}\, N = \frac{m}{4} + \sum_{t=1}^{m} \frac{t^2-1}{16} = \frac{3m}{16} + \frac{m(m+1)(2m+1)}{96} = \frac{m^3}{48} + \frac{m^2}{32} + \frac{19m}{96}.$$

Critically $\mathbf{Var}\, N$ is cubic in m, not quartic as one might naively expect. To simplify calculations, we use the upper bound $m^3/48 + m^2/32 + 19m/96 \le m^3/48 + 4m^3/96 = m^3/16$ for $m \ge 3$ to get $\mathbf{Var}\, N \le m^3/16$.

Lower bound. The concentration of measure estimate in Chebyshev's inequality

$$\mathbf{P}\Big[\, \big|Z - \mathbf{E}[Z]\big| \ge d\sqrt{\mathbf{Var}\, Z}\,\Big] \le \frac{1}{d^2}$$

implies that

$$\mathbf{P}\Big[\, \big|N - \frac{m(m+3)}{8}\big| \ge d\frac{m^{3/2}}{4}\Big] \le \frac{1}{d^2}$$

and so, taking the complementary event,

$$\mathbf{P}\Big[\, \big|N - \frac{m(m+3)}{8}\big| < d\frac{m^{3/2}}{4}\Big] \ge 1 - \frac{1}{d^2} \tag{6}$$

for $m \ge 3$ and any $d > 0$. This is the rigorous version of (2) in the informal overview.

We now make the critical step, *from probability to counting*. The probability space Ω consists of 2^m coin flip sequences. Since different coin flip sequences define different partitions, (6) implies that the proportion of coin flip sequences ω whose partition $Q(\omega)$ has size $N(\omega)$ with $\big|N(\omega) - m(m+3)/8\big| < dm^{3/2}/4$ is at least $1 - 1/d^2$. Hence we have the analogue of (3) in the informal overview:

$$\sum_n p(n) \ge 2^m\Big(1 - \frac{1}{d^2}\Big) \tag{7}$$

where the sum is over all $n \in \mathbf{N}$ such that $|n - m(m+3)/8| < dm^{3/2}/4$. Since $p(n)$ is positive increasing, we deduce, by the same argument used in the informal overview, that, for $m \ge 3$,

$$2\frac{dm^{3/2}}{4}p\Big(\frac{m(m+3)}{8} + d\frac{m^{3/2}}{4}\Big) \ge 2^m\Big(1 - \frac{1}{d^2}\Big)$$

where we extend the domain of p to the real numbers $\mathbf{R}$ by setting $p(x) = p(\lfloor x \rfloor)$. Hence

$$p\Big(\frac{m(m+3)}{8} + d\frac{m^{3/2}}{4}\Big) \geq \frac{2^m}{dm^{3/2}} 2\Big(1 - \frac{1}{d^2}\Big),$$

The function $d \mapsto \frac{1}{d}\big(1 - \frac{1}{d^2}\big)$ is maximized when $d = \sqrt{3}$, where it has value $\frac{2}{3\sqrt{3}}$. Therefore we take $d = \sqrt{3}$. Let $\eta > 0$ be given. Provided m is sufficiently large we have $3m/8 + \sqrt{3}m^{3/2}/4 < \eta m^2/8$. Hence

$$p\Big(\frac{m^2}{8}(1+\eta)\Big) \geq 2^m \frac{4}{3\sqrt{3}m^{3/2}} \tag{8}$$

for all m sufficiently large. Setting $n = m^2(1+\eta)/8$ and taking logs we obtain

$$\log p(n) \geq \sqrt{\frac{8n}{1+\eta}} \log 2 - \frac{3}{2} \log \frac{8n}{1+\eta} + \log \frac{4}{3\sqrt{3}}$$

for all n sufficiently large. Since $(\log n)/\sqrt{n} \to 0$ as $n \to \infty$ it follows that for each $\epsilon > 0$,

$$\frac{\log p(n)}{\sqrt{n}} > \frac{\sqrt{8}\log 2}{1+\epsilon}$$

for all n sufficiently large, as claimed in (1). The constant on the right-hand side is approximately 1.961, somewhat lower than the asymptotically correct $2\sqrt{\pi^2/6} \approx 2.565$. For a concrete lower bound, take $\eta = \frac{1}{8}$ and $m = 8\sqrt{n}/3$ in (8) to get $p(n) \geq 2^{8\sqrt{n}/3}/2^{5/2}n^{3/4}$ for all n sufficiently large. (It is easily seen that $n \geq 10^6$ suffices.) Using a computer to check small cases one can show that in fact

$$p(n) \geq \frac{2^{8\sqrt{n}/3}}{2^{5/2}n^{3/4}} \quad \text{for all } n \geq 2.$$

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