FOULKES MODULES AND DECOMPOSITION NUMBERS OF THE SYMMETRIC GROUP

EUGENIO GIANNELLI AND MARK WILDON

Abstract. The decomposition matrix of a finite group in prime characteristic p records the multiplicities of its p-modular irreducible representations as composition factors of the reductions modulo p of its irreducible representations in characteristic zero. The main theorem of this paper gives a combinatorial description of certain columns of the decomposition matrices of symmetric groups in odd prime characteristic. The result applies to blocks of arbitrarily high weight. It is obtained by studying the p-local structure of certain twists of the permutation module given by the action of the symmetric group of even degree $2m$ on the collection of set partitions of a set of size $2m$ into m sets each of size two. In particular, the vertices of the indecomposable summands of all such modules are characterized; these summands form a new family of indecomposable p-permutation modules for the symmetric group. As a further corollary it is shown that for every natural number w there is a diagonal Cartan number in a block of the symmetric group of weight w equal to $w + 1$.

1. Introduction

A central open problem in the representation theory of finite groups is to find the decomposition matrices of symmetric groups. The main result of this paper gives a combinatorial description of certain columns of these matrices in odd prime characteristic. This result applies to blocks of arbitrarily high weight. Another notable feature is that it is obtained almost entirely by using the methods of local representation theory.

We use the definitions of James' lecture notes [26] in which the rows of the decomposition matrix of the symmetric group S_n of degree n in prime characteristic p are labelled by the partitions of n, and the columns by the p-regular partitions of n, that is, partitions of n with at most $p-1$ parts of any given size. The entry $d_{\mu\nu}$ of the decomposition matrix records the number of composition factors of the Specht module S^μ, defined over a field of characteristic p, that are isomorphic to the simple module D^ν, first defined by James in [22] as the unique top composition factor of S^ν.

Given an odd number p, a p-core γ and $k \in \mathbf{N}_0$, let $w_k(\gamma)$ denote the minimum number of p-hooks that when added to γ give a partition with

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exactly k odd parts. (We recall the definition of p-hooks, p-cores and weights in §2 below.) Let $\mathcal{E}_k(\gamma)$ denote the set of partitions with exactly k odd parts that can be obtained from γ by adding $w_k(\gamma)$ disjoint p-hooks. Our main theorem is as follows.

Theorem 1.1. *Let p be an odd prime. Let γ be a p-core and let $k \in \mathbf{N}_0$. Let $n = |\gamma| + w_k(\gamma)p$. If $k \geq p$ suppose that*

$$w_{k-p}(\gamma) \neq w_k(\gamma) - 1.$$

Then $\mathcal{E}_k(\gamma)$ is equal to the disjoint union of subsets $\mathcal{X}_1, \ldots, \mathcal{X}_c$ such that each $\mathcal{X}_j$ has a unique maximal partition ν_j in the dominance order. Each ν_j is p-regular and the column of the decomposition matrix of S_n in characteristic p labelled by ν_j has 1s in the rows labelled by partitions in $\mathcal{X}_j$, and 0s in all other rows.

We leave it as a simple exercise to show that $w_k(\gamma)$ is well-defined. It may clarify the main hypothesis in Theorem 1.1 to remark that since $w_k(\gamma) \leq w_{k-p}(\gamma)+1$, we have $w_{k-p}(\gamma) \neq w_k(\gamma)-1$ if and only if $w_{k-p}(\gamma) > w_k(\gamma)-1$.

In particular Theorem 1.1 implies that if λ is a maximal partition in $\mathcal{E}_k(\gamma)$ under the dominance order, then the only non-zero entries of the column of the decomposition matrix labelled by λ are 1s in rows labelled by partitions in $\mathcal{E}_k(\gamma)$. We give some examples of Theorem 1.1 in Example 6.2.

Much of the existing work on decomposition matrices of symmetric groups has concentrated on giving complete information about blocks of small weight. In contrast, Theorem 1.1 gives partial information about blocks of arbitrary weight. In Proposition 6.4 we show that there are blocks of every weight in which Theorem 1.1 completely determines a column of the decomposition matrix.

We prove Theorem 1.1 by studying certain twists by the sign character of the permutation module $H^{(2^m)}$ given by the action of S_{2m} on the collection of all set partitions of $\{1, \ldots, 2m\}$ into m sets each of size two, defined over a field F. (Equivalently $H^{(2^m)}$ is the FS_{2m}-module induced from the trivial module for the imprimitive wreath product $S_2 \wr S_m \leq S_{2m}$.) For m, $k \in \mathbf{N}_0$, let

$$H^{(2^m\,;\,k)} = \big(H^{(2^m)} \boxtimes \mathrm{sgn}_{S_k}\big)\big\uparrow_{S_{2m}\times S_k}^{S_{2m+k}}$$

where $\boxtimes$ denotes the outer tensor product of two modules. Thus when $k=0$ we have $H^{(2^m\,;\,k)} = H^{(2^m)}$, and when $m = 0$ we have $H^{(2^m\,;\,k)} = \mathrm{sgn}_{S_k}$; if $k = m = 0$ then $H^{(2^m\,;\,k)}$ should be regarded as the trivial module for the trivial group S_0. It is known that the ordinary characters of these modules are multiplicity-free (see Lemma 3.1), but as one might expect, when F has prime characteristic, their structure can be quite intricate. Our main contribution is Theorem 1.2 below, which characterizes the vertices of indecomposable summands of $H^{(2^m\,;\,k)}$ when F has odd characteristic. The outline of the proof of Theorem 1.1 given at the end of this introduction shows how the local information given by Theorem 1.2 is translated into our result on decomposition matrices. This step, from local to global, is the key to the argument.

Theorem 1.2. *Let $m \in \mathbf{N}$ and let $k \in \mathbf{N}_0$. If U is an indecomposable non-projective summand of $H^{(2^m\,;\,k)}$, defined over a field F of odd characteristic p, then U has as a vertex a Sylow p-subgroup Q of $(S_2 \wr S_{tp}) \times S_{(r-2t)p}$ for some $t \in \mathbf{N}_0$ and $r \in \mathbf{N}$ with $tp \leq m$, $2t \leq r$ and $(r-2t)p \leq k$. Moreover the Green correspondent of U admits a tensor factorization $V \boxtimes W$ as a module for $F\big((N_{S_{rp}}(Q)/Q) \times S_{2m+k-rp}\big)$, where V and W are projective, and W is an indecomposable summand of $H^{(2^{m-tp}\,;\,k-(r-2t)p)}$.*

Theorem 1.2 is a significant result in its own right. For odd primes p, it gives the first infinite family of indecomposable p-permutation modules for the symmetric group (apart from Scott modules, which always lie in principal blocks) whose vertices are not Sylow p-subgroups of Young subgroups of symmetric groups.

An important motivation for the proof of Theorem 1.2 is [10], in which Erdmann uses similar methods to determine the p-local structure of Young permutation modules and to establish their decomposition into Young modules. Also relevant is [34], in which Paget shows that $H^{(2^m)}$ has a Specht filtration for any field F. Using Theorem 11 of [44], it follows that $H^{(2^m\,;\,k)}$ has a Specht filtration for every $k \in \mathbf{N}_0$. The local behaviour of $H^{(2^m)}$ in characteristic 2, which as one would expect is very different to the case of odd characteristic, was analysed in [8]; the projective summands of $H^{(2^m\,;\,k)}$ in characteristic 2 are identified in [33, Corollary 9]. In characteristic zero, the module $H^{(2^m)}$ arises in the first non-trivial case of Foulkes' Conjecture (see [17]). For this reason we call $H^{(2^m)}$ a *Foulkes module* and $H^{(2^m\,;\,k)}$ a *twisted Foulkes module*. For some recent results on the characters of general Foulkes modules we refer the reader to [18] and [35].

Background on decomposition numbers. The problem of finding decomposition numbers for symmetric groups in prime characteristic has motivated many deep results relating the representation theory of symmetric groups to other groups and algebras. Given the depth of the subject we give only a very brief survey, concentrating on results that apply to Specht modules in blocks of arbitrarily high weight.

Fix an infinite field F of prime characteristic p. In [29] James proved that the decomposition matrix for S_n modulo p appears, up to a column reordering, as a submatrix of the decomposition matrix for polynomial representations of $\mathrm{GL}_d(F)$ of degree n, for any $d \geq n$. In [19, 6.6g] Green gave an alternative proof of this using the Schur functor from representations of the Schur algebra to representations of symmetric groups. James later established a similar connection with representations of the finite groups $\mathrm{GL}_d(\mathbf{F}_q)$, and the Hecke algebras $\mathcal{H}_{F,q}(S_n)$, in the case when p divides $q-1$ (see [27]). In [9] Erdmann proved, conversely, that every decomposition number for $\mathrm{GL}_d(F)$ appears as an explicitly determined decomposition number for some symmetric group.

In [22] James proved that if D^ν is a composition factor of S^μ then ν dominates μ, and that if μ is p-regular then $d_{\mu\mu} = 1$. This establishes the characteristic 'wedge' shape of the decomposition matrix of S_n with 1s on its diagonal, shown in the diagram in [26, Corollary 12.3]. In [36] Peel proved that the hook Specht modules $(n-r, 1^r)$ are irreducible when p does not divide n, and described their composition factors for odd primes p when p divides n. The p-regular partitions labelling these composition factors can be determined by James' method of p-regularization [24], which gives for each partition μ of n a p-regular partition ν such that ν dominates μ and $d_{\mu\nu} = 1$. In [23] and [25], James determined the decomposition numbers $d_{\mu\nu}$ for μ of the form $(n-r, r)$ and, when $p = 2$, of the form $(n-r-1, r, 1)$. These results were extended by Williams in [46]. In [28, 5.47] James and Mathas, generalizing a conjecture of Carter, conjectured a necessary and sufficient condition on a partition μ for the Specht module S^μ, defined for a Hecke algebra $\mathcal{H}_{K,q}(S_n)$ over a field K, to be irreducible. The necessity of this condition was proved by Fayers [12] for symmetric groups (the case $q = 1$), building on earlier work of Lyle [31]; later Fayers [13] proved that the condition was sufficient for symmetric groups, and also for Hecke algebras whenever K has characteristic zero. In [30, Theorem 1.10], Kleshchev determined the decomposition numbers $d_{\lambda\mu}$ when μ is a p-regular partition whose Young diagram is obtained from the Young diagram of λ by moving a single box. In [43] the second author proved that in odd characteristic the rows of any decomposition matrix of a symmetric group are distinct, and so a Specht module is determined, up to isomorphism, by its multiset of composition factors; in characteristic 2 the isomorphism $(S^\mu)^\star = S^{\mu'}$, where μ' is the conjugate partition to μ, accounts for all pairs of equal rows in the decomposition matrix.

In [15] Fayers proved that the decomposition numbers in blocks of weight 3 of abelian defect are either 0 or 1. This paper includes a valuable summary of the many techniques for computing decomposition numbers and references to earlier results on blocks of weights 1 and 2. For results on weight 3 blocks of non-abelian defect, and blocks of weight 4, the reader is referred to [16] and [14]. For further general results, including branching rules and row and column removals theorems, see [32, Chapter 6, Section 4].

Outline. The main tool used to analyse the structure of twisted Foulkes modules over fields of odd characteristic is the Brauer correspondence for p-permutation modules, as developed by Broué in [6]. We state the necessary background results on the Brauer correspondence and blocks of symmetric groups in §2.

In §3 we collect the general results we need on twisted Foulkes modules. In particular, Lemma 3.1 gives their ordinary characters. The twisted Foulkes modules $H^{(2^m\,;\,k)}$ are p-permutation modules, but not permutation modules (except when $k \leq 1$), and so some care is needed when applying the Brauer correspondence. Our approach is to use Lemma 3.3 to construct explicit p-permutation bases: for more theoretical results on monomial modules for finite groups the reader is referred to [4].

The main part of the proof begins in §4 where we prove Theorem 1.2. In §5 we prove Theorem 1.1, by filling in the details in the following sketch. The hypotheses of Theorem 1.1, together with Lemma 3.1 on the ordinary character of $H^{(2^m\,;\,k)}$, imply that $H^{(2^m\,;\,k)}$ has a summand in the block of S_{2m+k} with p-core γ. If this summand is non-projective, then it follows from Theorem 1.2, using Theorem 2.7 on the Brauer correspondence between blocks of symmetric groups, that either $H^{(2^m\,;\,k-p)}$ has a summand in the block of S_{2m+k-p} with p-core γ, or one of $H^{(2^{m-p}\,;\,k)}$ and $H^{(2^m\,;\,k-2p)}$ has a summand in the block of $S_{2m+k-2p}$ with p-core γ. All of these are shown to be ruled out by the hypotheses of Theorem 1.1. Hence the summand is projective. A short argument using Lemma 3.1, Brauer reciprocity and Scott's lifting theorem then gives Theorem 1.1. We also obtain the proposition below, which identifies a particular projective summand of $H^{(2^m\,;\,k)}$ in the block of S_{2m+k} with p-core γ.

Proposition 1.3. *Let p be an odd prime, let γ be a p-core and let $k \in \mathbf{N}_0$. If $k \geq p$ suppose that $w_{k-p}(\gamma) \neq w_k(\gamma) - 1$. Let $2m + k = |\gamma| + w_k(\gamma)p$. If λ is a maximal partition in the dominance order on $\mathcal{E}_k(\gamma)$ then λ is p-regular and the projective cover of the simple module D^λ is a direct summand of $H^{(2^m\,;\,k)}$, where both modules are defined over a field of characteristic p.*

In §6 we give some further examples and corollaries of Theorem 1.1 and Proposition 1.3. In Lemma 6.3 we show that given any odd prime p, any $k \in \mathbf{N}_0$, and any $w \in \mathbf{N}$, there is a p-core γ such that $w_k(\gamma) = w$. We use these p-cores to show that the lower bound $c_{\lambda\lambda} \geq w + 1$ on the diagonal Cartan numbers in a block of weight w, proved independently by Richards [37, Theorem 2.8] and Bessenrodt and Uno [3, Proposition 4.6(i)], is attained for every odd prime p in p-blocks of every weight. Since the endomorphism algebra of each $H^{(2^m\,;\,k)}$ is commutative (in any characteristic), it also follows that for any odd prime p and any $w \in \mathbf{N}$, there is a projective module for a symmetric group lying in a p-block of weight w whose endomorphism algebra is commutative.

2. Preliminaries on the Brauer correspondence

In this section we summarize the principal results from [6] on the Brauer correspondence for p-permutation modules. We then recall some key facts on the blocks of symmetric groups. For background on vertices, sources and blocks we refer the reader to [1]. Throughout this section let F be a field of prime characteristic p.

The Brauer morphism. Let G be a finite group. An FG-module V is said to be a *p-permutation* module if whenever P is a p-subgroup of G, there exists an F-basis $\mathcal{B}$ of V whose elements are permuted by P. We say that $\mathcal{B}$ is a *p-permutation basis* of V with respect to P, and write $V = \langle \mathcal{B} \rangle$. It is easily seen that if V has a p-permutation basis with respect to a Sylow p-subgroup P of G then V is a p-permutation module.

The following proposition characterizing p-permutation modules is proved in [6, (0.4)]. As usual, if V and W are FG-modules we write $V \mid W$ to indicate that V is isomorphic to a direct summand of W.

Proposition 2.1. *An indecomposable FG-module V is a p-permutation module if and only if there exists a p-subgroup $P \leq G$ such that $V \mid F\!\uparrow_P^G$.*

Thus an indecomposable FG-module is a p-permutation module if and only if it has trivial source. It follows that the restriction or induction of a p-permutation module is still p-permutation, as is any summand of a p-permutation module.

We now recall the definition of the Brauer correspondence for general FG-modules before specializing to p-permutation modules. Let V be an FG-module. Given a p-subgroup $Q \leq G$ we let V^Q be the set $\{v \in V : vg = v \text{ for all } g \in Q\}$ of Q-fixed elements. It is easy to see that V^Q is an $FN_G(Q)$-module on which Q acts trivially. For R a proper subgroup of Q, the *relative trace* map $\mathrm{Tr}_R^Q : V^R \to V^Q$ is the linear map defined by

$$\mathrm{Tr}_R^Q(v) = \sum_g vg,$$

where the sum is over a set of right-coset representatives for R in Q. We observe that

$$\sum_{R<Q} \mathrm{Tr}_R^Q(V^R)$$

is an $FN_G(Q)$-module contained in V^Q. The *Brauer correspondent* of V with respect to Q is the $FN_G(Q)$-module $V(Q)$ defined by

$$V(Q) = V^Q / \sum_{R<Q} \mathrm{Tr}_R^Q(V^R).$$

It follows immediately from the definition of the Brauer correspondence that if U is another FG-module then $(U \oplus V)(Q) = U(Q) \oplus V(Q)$.

The following theorem is proved in [6, 3.2(1)].

Theorem 2.2. *Let V be an indecomposable p-permutation FG-module and let Q be a vertex of V. Let R be a p-subgroup of G. Then $V(R) \neq 0$ if and only if $R \leq Q^g$ for some $g \in G$.*

If V is an FG-module with p-permutation basis $\mathcal{B}$ with respect to a Sylow p-subgroup P of G and $R \leq P$, then taking for each orbit of R on $\mathcal{B}$ the sum of the vectors in that orbit, we obtain a basis for V^R. The sums over vectors lying in orbits of size p or more are relative traces from proper subgroups of R, and so $V(R)$ is equal to the F-span of

$$\mathcal{B}^R = \{v \in \mathcal{B} : vg = v \text{ for all } g \in R\}.$$

Thus Theorem 2.2 has the following corollary, which we shall use throughout §4.

Corollary 2.3. *Let V be a p-permutation FG-module with p-permutation basis $\mathcal{B}$ with respect to a Sylow p-subgroup P of G. Let $R \leq P$. The $FN_G(R)$-module $V(R)$ is equal to $\langle \mathcal{B}^R \rangle$ and V has an indecomposable summand with a vertex containing R if and only if $\mathcal{B}^R \neq \varnothing$.*

The following proposition shows that the Brauer correspondent of a p-permutation module is again a p-permutation module. This remark will be crucial in the proof of Theorem 1.2.

Proposition 2.4. *Let U be a p-permutation FG-module and let R be a p-subgroup of G. The Brauer correspondent $U(R)$ of U is a p-permutation $FN_G(R)$-module.*

Proof. Let P' be a Sylow p-subgroup of $N_G(R)$ and let P be a Sylow p-subgroup of G containing P'. Let $\mathcal{B}$ be a p-permutation basis of U with respect to P. By Corollary 2.3 the $FN_G(R)$-module $U(R)$ has $\mathcal{B}^R$ as a basis. Since $R \leq N_G(P')$, it follows that $\mathcal{B}^R$ is a p-permutation basis of $U(R)$ with respect to P'. Therefore $U(R)$ is a p-permutation $FN_G(R)$-module. □

Remarkably, the Brauer correspondent of an indecomposable p-permutation module with respect to its vertex agrees with its Green correspondent. This is proved in [41, Exercise 27.4]. We therefore have the following theorem (see [6, 3.6]).

Theorem 2.5. *The Brauer correspondence sending V to $V(Q)$ is a bijection between the set of indecomposable p-permutation FG-modules with vertex Q and the set of indecomposable projective $FN_G(Q)/Q$-modules. The $FN_G(Q)$-module $V(Q)$ is the Green correspondent of V.*

Blocks of the symmetric group. The blocks of symmetric groups are described combinatorially by Nakayama's Conjecture, first proved by Brauer and Robinson in two connected papers [38] and [5]. In order to state this result, we must recall some definitions.

Let λ be a partition. A *p-hook* in λ is a connected part of the rim of the Young diagram of λ consisting of exactly p boxes, whose removal leaves the diagram of a partition. By repeatedly removing p-hooks from λ we obtain the *p-core* of λ; the number of hooks we remove is the *weight* of λ.

Theorem 2.6 (Nakayama's Conjecture)**.** *Let p be prime. The p-blocks of S_n are labelled by pairs (γ, w), where γ is a p-core and $w \in \mathbf{N}_0$ is the associated weight, such that $|\gamma| + wp = n$. Thus the Specht module S^λ lies in the block labelled by (γ, w) if and only if λ has p-core γ and weight w.* $\square$

We denote the block of weight w corresponding to the p-core γ by $B(\gamma, w)$. The following description of the Brauer correspondence for blocks of symmetric groups is critical to the proof of Proposition 5.1 below.

Theorem 2.7. *Let V be an indecomposable p-permutation module lying in the block $B(\gamma, w)$ of S_n. Suppose that R is contained in a vertex of V and that R moves exactly the first rp elements of $\{1, \ldots, n\}$; that is* $\mathrm{supp}\,(R) = \{1, \ldots, rp\}$. *Then $N_{S_n}(R) \cong N_{S_{rp}}(R) \times S_{n-rp}$. Moreover,*

(i) *$N_{S_{rp}}(R)$ has a unique block, b say.*
(ii) *The blocks $b \otimes B(\gamma, w-r)$ and $B(\gamma, w)$ are Brauer correspondents.*
(iii) *As an $FN_{S_n}(R)$-module, $V(R)$ lies in $b \otimes B(\gamma, w-r)$.*

Proof. Part (i) is an immediate corollary of Lemma 2.6 and the following sentence of [7]. Part (ii) is stated in (2) on page 166 of [7], and then proved as a corollary of the characterisation of maximal Brauer pairs given in Proposition 2.12 of [7]. Part (iii) follows from Lemma 7.4 of [45]. $\square$

3. Foulkes modules and twisted Foulkes modules

Throughout this section let F be a field and let $m \in \mathbf{N}$, $k \in \mathbf{N}_0$. We define $\Omega^{(2^m)}$ to be the collection of all set partitions of $\{1, \ldots, 2m\}$ into m sets each of size two. The symmetric group S_{2m} acts on $\Omega^{(2^m)}$ in an obvious way. We have already defined the Foulkes module $H^{(2^m)}$ to be the permutation module with F-basis $\Omega^{(2^m)}$, and the twisted Foulkes module $H^{(2^m\,;k)}$ to be $\big(H^{(2^m)} \boxtimes \mathrm{sgn}_{S_k}\big)\big\uparrow_{S_{2m} \times S_k}^{S_{2m+k}}$.

Let χ^λ denote the irreducible character of S_n corresponding to the partition λ of n. When F has characteristic zero the ordinary character of $H^{(2^m)}$ was found by Thrall [42, Theorem III] to be $\sum_\mu \chi^{2\mu}$ where the sum is over all partitions μ of m and 2μ is the partition obtained from μ by doubling each part.

Lemma 3.1. *The ordinary character of* $H^{(2^m\,;\,k)}$ *is* $\sum_\lambda \chi^\lambda$*, where the sum is over all partitions* λ *of* $2m+k$ *with exactly* k *odd parts.*

Proof. This follows from Pieri's rule (see [40, 7.15.9]) applied to the ordinary character of $H^{(2^m)}$. □

We remark that an alternative proof of Lemma 3.1 with minimal prerequisites can be found in [20]; the main result of [20] uses the characters of twisted Foulkes modules to construct a 'model' character for each symmetric group containing each irreducible character exactly once.

In the remainder of this section we suppose that F has odd characteristic p and define a module isomorphic to $H^{(2^m\,;\,k)}$ that will be used in the calculations in §4. Let S_X denote the symmetric group on the set X. Let $\Delta^{(2^m\,;\,k)}$ be the set of all elements of the form

$$\big\{\{i_1,i'_1\},\ldots,\{i_m,i'_m\},(j_1,\ldots,j_k)\big\}$$

where $\{i_1,i'_1,\ldots,i_m,i'_m,j_1,\ldots,j_k\}=\{1,\ldots,2m+k\}$. Given $\delta\in\Delta^{(2^m\,;\,k)}$ of the form above, we define

$$\begin{aligned}\mathcal{S}(\delta)&=\big\{\{i_1,i'_1\},\ldots,\{i_m,i'_m\}\big\},\\ \mathcal{T}(\delta)&=\{j_1,\ldots,j_k\}.\end{aligned}$$

The symmetric group S_{2m+k} acts transitively on $\Delta^{(2^m\,;\,k)}$ by

$$\delta g=\big\{\{i_1g,i'_1g\},\ldots,\{i_mg,i'_mg\},(j_1g,\ldots,j_kg)\big\}$$

for $g\in S_{2m+k}$. Let $F\Delta^{(2^m\,;\,k)}$ be the permutation module for FS_{2m+k} with F-basis $\Delta^{(2^m\,;\,k)}$. Let $K^{(2^m\,;\,k)}$ be the subspace of $F\Delta^{(2^m\,;\,k)}$ spanned by

$$\big\{\delta-\operatorname{sgn}(g)\delta g:\delta\in\Delta^{(2^m\,;\,k)},g\in S_{\mathcal{T}(\delta)}\big\}.$$

Since this set is permuted by S_{2m+k}, it is clear that $K^{(2^m\,;\,k)}$ is an FS_{2m+k}-submodule of $F\Delta^{(2^m\,;\,k)}$. For $\delta\in\Delta^{(2^m\,;\,k)}$, let $\overline{\delta}\in F\Delta^{(2^m\,;\,k)}/K^{(2^m\,;\,k)}$ denote the image $\delta+K^{(2^m\,;\,k)}$ of δ under the quotient map. Let $\Omega^{(2^m\,;\,k)}$ be the subset of $\Delta^{(2^m\,;\,k)}$ consisting of those elements of the form above such that $j_1<\ldots<j_k$. In the next lemma we use $\Omega^{(2^m\,;\,k)}$ to identify $F\Delta^{(2^m\,;\,k)}/K^{(2^m\,;\,k)}$ with $H^{(2^m\,;\,k)}$.

Lemma 3.2.

(i) *For each* $\delta\in\Delta^{(2^m\,;\,k)}$ *there exists a unique* $\omega\in\Omega^{(2^m\,;\,k)}$ *such that* $\overline{\delta}\in\{\overline{\omega},-\overline{\omega}\}$*. Moreover, for this* ω *we have* $\mathcal{S}(\delta)=\mathcal{S}(\omega)$ *and* $\mathcal{T}(\delta)=\mathcal{T}(\omega)$ *and there exists a unique* $h\in S_{\mathcal{T}(\delta)}$ *such that* $\delta h=\omega$*.*

(ii) *The set* $\{\overline{\omega}:\omega\in\Omega^{(2^m\,;\,k)}\}$ *is an* F*-basis for* $F\Delta^{(2^m\,;\,k)}/K^{(2^m\,;\,k)}$*.*

(iii) *The* FS_{2m+k}*-modules* $H^{(2^m\,;\,k)}$ *and* $F\Delta^{(2^m\,;\,k)}/K^{(2^m\,;\,k)}$ *are isomorphic.*

Proof. For brevity we write K for $K^{(2^m\,;\,k)}$. Let

$$\delta = \big\{\{i_1, i_1'\}, \ldots, \{i_m, i_m'\}, (j_1, \ldots, j_k)\big\} \in \Delta^{(2^m\,;\,k)}.$$

The unique $h \in S_{\mathcal{T}(\delta)}$ such that $\delta h \in \Omega^{(2^m\,;\,k)}$ is the permutation such that $j_1 h < \ldots < j_k h$. Set $\omega = \delta h$. Since $S(\omega) = S(\delta)$ and $T(\omega) = T(\delta)$ we have the existence part of (i). Since $\delta - \mathrm{sgn}(h)\delta h \in K$, it follows that $\overline{\delta} = \mathrm{sgn}(h)\overline{\omega}$, and that $F\Delta^{(2^m\,;\,k)}/K$ is spanned by $\{\overline{\omega} : \omega \in \Omega^{(2^m\,;\,k)}\}$. Set $x = h^{-1}$. If $g \in S_{\mathcal{T}(\delta)}$ then

$$\delta - \mathrm{sgn}(g)\delta g = -\,\mathrm{sgn}(x)\big(\omega - \mathrm{sgn}(x)\omega x\big) + \mathrm{sgn}(x)\big(\omega - \mathrm{sgn}(xg)\omega x g\big).$$

Since $\mathcal{T}(\delta) = \mathcal{T}(\omega)$ we have x, $xg \in S_{\mathcal{T}(\omega)}$. It follows that K is spanned by

$$\{\omega - \mathrm{sgn}(y)\omega y : \omega \in \Omega^{(2^m\,;\,k)}, y \in S_{\mathcal{T}(\omega)}\}.$$

Hence $\dim F\Delta^{(2^m\,;\,k)}/K \le |\Omega^{(2^m\,;\,k)}|$ and $\dim K \le |\Omega^{(2^m\,;\,k)}|(k! - 1)$. Since $\dim F\Delta^{(2^m\,;\,k)} = |\Omega^{(2^m\,;\,k)}|k!$ we have equality in both cases. This proves part (ii). Moreover, if $\overline{\omega} = \pm\overline{\omega'}$ for $\omega, \omega' \in \Omega^{(2^m\,;\,k)}$ then $S(\omega) = S(\omega')$ and $T(\omega) = T(\omega')$, and so $\omega = \omega'$. This proves the uniqueness in (i).

For (iii), let

$$\omega = \big\{\{1,2\}, \ldots, \{2m-1, 2m\}, (2m+1, \ldots, 2m+k)\big\} \in \Omega^{(2^m\,;\,k)}.$$

Write $S_{2m} \times S_k$ for $S_{\{1,\ldots,2m\}} \times S_{\{2m+1,\ldots,2m+k\}}$, thought of as a subgroup of S_{2m+k}. Given $h \in S_{\{1,\ldots,2m\}}$ and $x \in S_{\{2m+1,\ldots,2m+k\}}$ we have $\overline{\omega}hx = \mathrm{sgn}(x)\overline{\omega h}$. By (ii) the set $\{\overline{\omega h} : h \in S_{2m}\}$ is linearly independent. Hence the $F(S_{2m} \times S_k)$-submodule of $F\Delta^{(2^m\,;\,k)}/K$ generated by $\overline{\omega}$ is isomorphic to $H^{(2^m)} \boxtimes \mathrm{sgn}_{S_k}$. Since

$$\dim F\Delta^{(2^m\,;\,k)}/K = |\Omega^{(2^m\,;\,k)}| = \binom{2m+k}{k} \dim H^{(2^m)}$$

and the index of $S_{2m} \times S_k$ in S_{2m+k} is $\binom{2m+k}{k}$, it follows that

$$F\Delta^{(2^m\,;\,k)}/K \cong \big(H^{(2^m)} \boxtimes \mathrm{sgn}_{S_k}\big)\big\uparrow_{S_{2m}\times S_k}^{S_{2m+k}}$$

as required. □

Since p is odd we have that

$$F\uparrow_{(S_2 \wr S_m)\times A_k}^{(S_2 \wr S_m)\times S_k} = F_{(S_2\wr S_m)\times S_k} \oplus \big(F_{S_2 \wr S_m} \boxtimes \mathrm{sgn}_{S_k}\big)$$

where A_k denotes the alternating group on $\{2k+1, \ldots, 2k+n\}$. Hence $H^{(2^m\,;\,k)}$ is a direct summand of the module induced from the trivial $F\big((S_2 \wr S_m) \times A_k\big)$-module, and so $H^{(2^m\,;\,k)}$ is a p-permutation module.

In the following lemma we construct a p-permutation basis for $H^{(2^m\,;\,k)}$.

Lemma 3.3. *Let P be a p-subgroup of S_{2m+k}.*

(i) *There is a choice of signs $s_\omega \in \{+1, -1\}$ for $\omega \in \Omega^{(2^m\,;\,k)}$ such that $\{s_\omega \overline{\omega} : \omega \in \Omega^{(2^m\,;\,k)}\}$ is a p-permutation basis for $H^{(2^m\,;\,k)}$ with respect to P.*

(ii) *Let* $\omega = \bigl\{\{i_1,i_1'\},\ldots,\{i_m,i_m'\},(j_1,\ldots,j_k)\bigr\} \in \Omega^{(2^m\,;\,k)}$ *and let* $g \in P$. *Then* $\overline{\omega}$ *is fixed by* g *if and only if* $\bigl\{\{i_1,i_1'\},\ldots,\{i_m,i_m'\}\bigr\}$ *is fixed by* g.

Proof. For $\omega \in \Omega^{(2^m\,;\,k)}$ let $\gamma(\omega) = \bigl(\mathcal{S}(\omega),\mathcal{T}(\omega)\bigr)$. Let

$$\Gamma^{(2^m\,;\,k)} = \bigl\{\gamma(\omega) : \omega \in \Omega^{(2^m\,;\,k)}\bigr\}.$$

Notice that the map

$$\gamma : \Omega^{(2^m;k)} \longrightarrow \Gamma^{(2^m\,;\,k)}$$

associating to each $\omega \in \Omega^{(2^m;k)}$ the element $\gamma(\omega) \in \Gamma^{(2^m\,;\,k)}$ is a bijection. The set $\Gamma^{(2^m\,;\,k)}$ is acted on by S_{2m+k} in the obvious way. Let $\gamma_1,\ldots,\gamma_c \in \Gamma^{(2^m\,;\,k)}$ be representatives for the orbits of P on $\Gamma^{(2^m\,;\,k)}$. For each $b \in \{1,\ldots,c\}$, let $\omega_b \in \Omega^{(2^m\,;\,k)}$ be the unique element such that $\gamma_b = \gamma(\omega_b)$. Given any $\omega \in \Omega^{(2^m\,;\,k)}$ there exists a unique b such that $\gamma(\omega)$ is in the orbit of P on $\Gamma^{(2^m\,;\,k)}$ containing γ_b. Choose $g \in P$ such that $\gamma(\omega) = \gamma_b g$. Then ω and $\omega_b g$ are equal up to the order of the numbers in their k-tuples, and so there exists $h \in S_{\mathcal{T}(\omega)}$ such that $\omega_b g h = \omega$. By Lemma 3.2(i) we have

$$\overline{\omega_b} g = s_\omega \overline{\omega}$$

for some $s_\omega \in \{+1,-1\}$. If $\tilde{g} \in P$ is another permutation such that $\gamma(\omega) = \gamma_b\tilde{g}$ then $\overline{\omega_b} g\tilde{g}^{-1} = \pm\overline{\omega_b}$. Hence the F-span of $\overline{\omega_b}$ is a 1-dimensional representation of the cyclic p-group generated by $g\tilde{g}^{-1}$. The unique such representation is the trivial one, so $\overline{\omega_b} g = \overline{\omega_b}\tilde{g}$. The sign s_ω is therefore well-defined. Now suppose that ω, $\omega' \in \Omega^{(2^m\,;\,k)}$ and $h \in P$ are such that $s_\omega\overline{\omega}h = \pm s_{\omega'}\overline{\omega'}$. By construction of the basis there exists $\omega_b \in \Omega^{(2^m\,;\,k)}$ and g, $g' \in P$ such that $s_\omega\overline{\omega} = \overline{\omega_b}g$ and $s_{\omega'}\overline{\omega'} = \overline{\omega_b}g'$. Therefore

$$\overline{\omega_b}gh = s_\omega\overline{\omega}h = \pm s_{\omega'}\overline{\omega'} = \pm\overline{\omega_b}g'$$

and so $\overline{\omega_b}gh{g'}^{-1} = \pm\overline{\omega_b}$. As before, the plus sign must be correct. This proves (i).

For (ii), suppose that $\overline{\omega}g = \overline{\omega}$. Setting $\delta = \omega g$, and noting that $\overline{\delta} = \overline{\omega}$, it follows from Lemma 3.2(i) that $\mathcal{S}(\omega g) = \mathcal{S}(\delta) = \mathcal{S}(\omega)$. Hence the condition in (ii) is necessary. Conversely, if $\bigl\{\{i_1,i_1'\},\ldots,\{i_m,i_m'\}\bigr\}$ is fixed by g then g permutes $\{j_1,\ldots,j_k\}$ and so $\overline{\omega}g \in \{\overline{\omega},-\overline{\omega}\}$. Since $g \in P$, it now follows from (i) that $\overline{\omega}g = \overline{\omega}$, as required. □

In applications of Lemma 3.3(ii) it will be useful to note that there is an isomorphism of S_{2m}-sets between $\Omega^{(2^m)}$ and the set of fixed-point free involutions in S_{2m}, where the symmetric group acts by conjugation. Given $\omega \in \Omega^{(2^m\,;\,k)}$ with $\mathcal{S}(\omega) = \bigl\{\{i_1,i_1'\},\ldots,\{i_m,i_m'\}\bigr\}$, we define

$$\mathcal{I}(\omega) = (i_1,i_1')\cdots(i_m,i_m') \in S_{2m+k}.$$

By Lemma 3.3(ii), if $g \in S_{2m+k}$ is a p-element then g fixes $\overline{\omega}$ if and only if g commutes with $\mathcal{I}(\omega)$. Corollary 2.3 and Lemma 3.3 therefore imply the following proposition, which we shall use repeatedly in the next section.

Proposition 3.4. *Let R be a p-subgroup of S_{2m+k} and let P be a Sylow p-subgroup of S_{2m+k} containing a Sylow p-subgroup of $N_G(R)$. There is a choice of signs $s_\omega \in \{+1, -1\}$ for $\omega \in \Omega^{(2^m\,;\,k)}$ such that*

$$\bigl\{s_\omega\overline{\omega} : \omega \in \Omega^{(2^m\,;\,k)},\, \mathcal{I}(\omega) \in \mathrm{C}_{S_{2m+k}}(R)\bigr\}.$$

is a p-permutation basis for the Brauer correspondent $H^{(2^m\,;\,k)}(R)$ with respect to $P \cap N_G(R)$.

4. The local structure of $H^{(2^m\,;\,k)}$

In this section we prove Theorem 1.2. Throughout we let F be a field of odd characteristic p and fix $m \in \mathbf{N}$, $k \in \mathbf{N}_0$. Any vertex of an indecomposable non-projective summand of $H^{(2^m\,;\,k)}$ must contain, up to conjugacy, one of the subgroups

$$R_r = \langle z_1 z_2 \cdots z_r \rangle$$

where z_j is the p-cycle $(p(j-1)+1, \ldots, pj)$ and $rp \le 2m+k$, so we begin by calculating $H^{(2^m\,;\,k)}(R_r)$. In the second step we show that, for any $t \in \mathbf{N}$ such that $2t \le r$, the Brauer correspondent $H^{(2^{tp}\,;\,(r-2t)p)}(R_r)$ is indecomposable as an $FN_{S_{rp}}(R_r)$-module and determine its vertex; in the third step we combine these results to complete the proof.

In the second step we shall require the basic lemma below; its proof is left to the reader.

Lemma 4.1. *If P is a p-group and Q is a subgroup of P then the permutation module $F{\uparrow}_Q^P$ is indecomposable with vertex Q.*

First step: Brauer correspondent with respect to R_r. Let $r \in \mathbf{N}$ be such that $rp \le 2m + k$. We define

$$T_r = \{t \in \mathbf{N}_0 : tp \le m,\ 2t \le r,\ (r-2t)p \le k\}.$$

For $t \in T_r$ let

$$\mathcal{A}_{2t} = \left\{\overline{\omega}\ :\ \begin{array}{l} \omega \in \Omega^{(2^m\,;\,k)}, \mathcal{I}(\omega) \in \mathrm{C}_{S_{2m+k}}(R_r) \\ \text{supp}\ \mathcal{I}(\omega) \text{ contains exactly } 2t \text{ orbits of } R_r \text{ of length } p \end{array}\right\}.$$

Lemma 4.2. *There is a direct sum decomposition of $FN_{S_{2m+k}}(R_r)$-modules*

$$H^{(2^m\,;\,k)}(R_r) \cong \bigoplus_{t \in T_r} \langle \mathcal{A}_{2t} \rangle.$$

Proof. By Proposition 3.4 the $FN_{S_{2m+k}}(R_r)$-module $H^{(2^m\,;\,k)}(R_r)$ has as a basis

$$\mathcal{A} = \{\overline{\omega} : \omega \in \Omega^{(2^m\,;\,k)},\ \mathcal{I}(\omega) \in \mathrm{C}_{S_{2m+k}}(R_r)\}.$$

Let $\omega \in \Omega^{(2^m\,;\,k)}$ be such that $\mathcal{I}(\omega) \in \mathrm{C}_{S_{2m+k}}(R_r)$. Then $\mathcal{I}(\omega)$ permutes, as blocks for its action, the orbits of R_r. It follows that the number of orbits of R_r of length p contained in supp $\mathcal{I}(\omega)$ is even. Suppose this number is $2t$. Clearly $2t \le r$ and $2tp \le 2m$. The remaining $r - 2t$ orbits of length p are contained in $\mathcal{T}(\omega)$. Thus $(r-2t)p \le k$, and so $t \in T_r$ and $\overline{\omega} \in \mathcal{A}_{2t}$.

Let the p-cycles corresponding to the $2t$ orbits of R_r contained in supp $\mathcal{I}(\omega)$ be $z_{j_1}, \ldots, z_{j_{2t}}$. Let $g \in N_{S_{2m+k}}(R_r)$. Let $\omega^\star \in \Omega^{(2^m\,;\,k)}$ be such that $\overline{\omega^\star} = \pm\,\overline{\omega g}$. The p-cycles $z_{j_1}^g, \ldots, z_{j_{2t}}^g$ correspond precisely to the orbits of R_r contained in supp $\mathcal{I}(\omega^\star)$. Hence $\overline{\omega^\star} \in \mathcal{A}_{2t}$, and so the vector space $\langle \mathcal{A}_{2t} \rangle$ is invariant under g. Since $\mathcal{A} = \bigcup_{t \in T_r} \mathcal{A}_{2t}$ the lemma follows. □

There is an obvious factorization $N_{S_{2m+k}}(R_r) = N_{S_{rp}}(R_r) \times S_{\{rp+1,\ldots,2m+k\}}$. The next proposition establishes a corresponding tensor factorization of the $N_{S_{2m+k}}(R_r)$-module $\langle \mathcal{A}_{2t} \rangle$. The shift required to make the second factor $H^{(2^{m-tp}\,;\,k-(r-2t)p)}$ a module for $FS_{\{rp+1,\ldots,2m+k\}}$ is made explicit in the proof.

Proposition 4.3. *If $t \in T_r$ then there is an isomorphism*

$$\langle \mathcal{A}_{2t} \rangle \cong H^{(2^{tp}\,;\,(r-2t)p)}(R_r) \boxtimes H^{(2^{m-tp}\,;\,k-(r-2t)p)}$$

of $F(N_{S_{rp}}(R_r) \times S_{\{rp+1,\ldots,2m+k\}})$-modules.

Proof. In order to simplify the notation we shall write K for the FS_{2m+k}-submodule $K^{(2^m\,;\,k)}$ of $F\Delta^{(2^m\,;\,k)}$ defined just before Lemma 3.2. Recall that if $\omega \in \Omega^{(2^m\,;\,k)}$ then, by definition, $\overline{\omega} = \omega + K$. Let $J = K^{(2^{tp}\,;\,(r-2t)p)}$. It follows from Proposition 3.4, in the same way as in Lemma 4.2, that $H^{(2^{tp}\,;\,(r-2t)p)}(R_r)$ has as a basis

$$\{\omega + J : \omega \in \Omega^{(2^{tp}\,;\,(r-2t)p)},\ \mathcal{I}(\omega) \in \mathrm{C}_{S_{rp}}(R_r)\}.$$

Define Δ^+ by shifting the entries in each of the elements of $\Delta^{(2^{m-tp},k-(r-2t)p)}$ by rp, so that $F\Delta^+$ is an $FS_{\{rp+1,\ldots,2m+k\}}$-module, and similarly define $\Omega^+ \subseteq \Delta^+$ by shifting $\Omega^{(2^{m-tp},k-(r-2t)p)}$ and $J^+ \subseteq F\Delta^+$ by shifting the basis elements of $K^{(2^{m-tp},k-(r-2t)p)}$. Then, by Lemma 3.2, $H^+ = F\Delta^+/J^+$ is an $FS_{\{rp+1,\ldots,2m+k\}}$-module with basis

$$\{\omega^+ + J^+ : \omega^+ \in \Omega^+\}.$$

We shall define a linear map $f : \langle \mathcal{A}_{2t} \rangle \to H^{(2^{tp}\,;\,(r-2t)p)} \boxtimes H^+$. Given $\omega + K \in \mathcal{A}_{2t}$ where

$$\omega = \{\{i_1, i_1'\}, \ldots, \{i_m, i_m'\}, (j_1, \ldots, j_k)\} \in \Omega^{(2^m\,;\,k)}$$

and the notation is chosen so that

$$\{i_1, i'_1, \ldots, i_{tp}, i'_{tp}, j_1, \ldots, j_{(r-2t)p}\} = \{1, \ldots, rp\},$$

we define $(\omega + K)f = (\alpha + J) \otimes (\alpha^+ + J^+)$ where

$$\begin{aligned} \alpha &= \bigl\{\{i_1, i'_1\}, \ldots, \{i_{tp}, i'_{tp}\}, (j_1, \ldots, j_{(r-2t)p})\bigr\} \\ \alpha^+ &= \bigl\{\{i_{tp+1}, i'_{tp+1}\}, \ldots, \{i_{mp}, i'_{mp}\}, (j_{(r-2t)p+1}, \ldots, j_k)\bigr\}. \end{aligned}$$

This defines a bijection between $\mathcal{A}_{2t}$ and the basis for $H^{(2^{tp}\,;\,(r-2t)p)}(R_r) \boxtimes H^+$ afforded by the bases for $H^{(2^{tp}\,;\,(r-2t)p)}(R_r)$ and H^+ just defined. The map f is therefore a well-defined linear isomorphism.

Suppose that $\omega \in \Omega^{(2^m\,;\,k)}$ is as above and let $g \in N_{S_{2m+k}}(R_r)$. Let $h \in S_{\mathcal{T}(\omega g)}$ be the unique permutation such that $(j_1 gh, \ldots, j_k gh)$ is increasing. Let $\omega^\star = \omega g h$, so $\omega^\star \in \Omega^{(2^m\,;\,k)}$ and $\overline{\omega} g = \mathrm{sgn}(h)\overline{\omega^\star}$. Since g permutes $\{1, \ldots, rp\}$ we may factorize h as $h = xx^+$ where $x \in S_{\mathcal{T}(\alpha)}$ and $x^+ \in S_{\mathcal{T}(\alpha^+)}$. By definition of f we have

$$(\omega^\star + K)f = (\alpha g x + J) \otimes (\alpha^+ g x^+ + J^+).$$

Hence

$$\begin{aligned} (\omega + K)gf &= \mathrm{sgn}(h)(\omega^\star + K)f \\ &= \mathrm{sgn}(h)\,\mathrm{sgn}(x)\,\mathrm{sgn}(x^+)(\alpha g + J) \otimes (\alpha^+ g + J^+) \\ &= (\omega + K)fg. \end{aligned}$$

The map f is therefore a homomorphism of $FN_{S_{2m+k}}(R_r)$-modules. Since f is a linear isomorphism, the proposition follows. □

Second step: the vertex of $H^{(2^{tp}\,;\,(r-2t)p)}(R_r)$. Fix $r \in \mathbf{N}$ and $t \in \mathbf{N}_0$ such that $2t \leq r$. In the second step we show that the $FN_{S_{rp}}(R_r)$-module $H^{(2^{tp}\,;\,(r-2t)p)}(R_r)$ is indecomposable and that it has the subgroup Q_t defined below as a vertex.

To simplify the notation, we denote $H^{(2^{tp}\,;\,(r-2t)p)}(R_r)$ by M. Let C and E_t be the elementary abelian p-subgroups of $N_{S_{rp}}(R_r)$ defined by

$$\begin{aligned} C &= \langle z_1 \rangle \times \langle z_2 \rangle \times \cdots \times \langle z_r \rangle, \\ E_t &= \langle z_1 z_{t+1} \rangle \times \cdots \times \langle z_t z_{2t} \rangle \times \langle z_{2t+1} \rangle \times \cdots \times \langle z_r \rangle, \end{aligned}$$

where the z_j are the p-cycles defined at the start of this section. For $i \in \{1, \ldots, tp\}$, let $i' = i + tp$, and for $g \in S_{\{1,\ldots,tp\}}$, let $g' \in S_{\{tp+1,\ldots,2tp\}}$ be the permutation defined by $i'g' = (ig)'$. Note that if $1 \leq j \leq t$ then $z'_j = z_{j+t}$. Let L be the group consisting of all permutations gg' where g lies in a Sylow p-subgroup of $S_{\{1,\ldots,tp\}}$ with base group $\langle z_1, \ldots, z_t \rangle$, chosen so that $z_1 \cdots z_t$ is in its centre. Let L^+ be a Sylow p-subgroup of $S_{\{2tp+1,\ldots,rp\}}$ with base group $\langle z_{2t+1}, \ldots, z_r \rangle$, chosen so that $z_{2t+1} \cdots z_r$ is in its centre. (The existence of such Sylow p-subgroups follows from the construction of Sylow p-subgroups of symmetric groups as iterated wreath products in [21, 4.1.19 and 4.1.20].) Let

$$Q_t = L \times L^+.$$

Observe that Q_t normalizes C and so $\langle C, Q_t\rangle$ is a p-group contained in $\mathrm{C}_{S_{rp}}(R_r)$. Let P be a Sylow p-subgroup of $\mathrm{C}_{S_{rp}}(R_r)$ containing $\langle Q_t, C\rangle$. Since there is a Sylow p-subgroup of S_{rp} containing R_r in its centre, P is also a Sylow p-subgroup of S_{rp}. Clearly $E_t \leq C$ and

$$R_r \leq E_t \leq Q_t \leq P \leq \mathrm{C}_{S_{rp}}(R_r).$$

If $t = 0$ then M is the sign representation of $N_{S_{rp}}(R_r)$, with p-permutation basis $\mathcal{B} = \{\omega\}$ where ω is the unique element of $\Omega^{(2^0\,;\,rp)}$. It is then clear that M has the Sylow p-subgroup Q_0 of $\mathrm{C}_{S_{rp}}(R_r)$ as a vertex. We may therefore assume that $t \in \mathbf{N}$ for the rest of this step.

By Proposition 3.4 there is a choice of signs $s_\omega \in \{+1, -1\}$ for $\omega \in \Omega^{(2^{tp}\,;\,(r-2t)p)}$ such that

$$\mathcal{B} = \{s_\omega \overline{\omega} : \omega \in \Omega^{(2^{tp}\,;\,(r-2t)p)},\ \mathcal{I}(\omega) \in \mathrm{C}_{S_{rp}}(R_r)\}$$

is a p-permutation basis for M with respect to P. Let

$$\mathcal{O}_j = \{(j-1)p + 1, \ldots, jp\}$$

be the orbit of z_j on $\{1, \ldots, rp\}$ of length p. If $\mathcal{I}(\omega) \in \mathrm{C}_{S_{rp}}(R_r)$ then $\mathcal{I}(\omega)$ permutes these orbits as blocks for its action; let

$$\mathcal{I}_{\mathcal{O}}(\omega) \in S_{\{\mathcal{O}_1, \ldots, \mathcal{O}_r\}}$$

be the involution induced by the action of $\mathcal{I}(\omega)$ on the set of orbits.

Proposition 4.4. *The $FN_{S_{rp}}(R_r)$-module M is indecomposable and has a vertex containing E_t.*

Proof. For each involution $h \in S_{\{\mathcal{O}_1, \ldots, \mathcal{O}_r\}}$ that fixes exactly $r - 2t$ of the orbits $\mathcal{O}_j$, and so moves the other $2t$, define

$$\mathcal{B}(h) = \{s_\omega \overline{\omega} \in \mathcal{B} : \mathcal{I}_{\mathcal{O}}(\omega) = h\}.$$

Clearly there is a vector space decomposition

$$M = \bigoplus_h \langle \mathcal{B}(h)\rangle.$$

If $g \in C$ then $\mathcal{I}_{\mathcal{O}}(\omega g) = \mathcal{I}_{\mathcal{O}}(\omega)$ since g acts trivially on the set of orbits $\{\mathcal{O}_1, \ldots, \mathcal{O}_r\}$. Therefore C permutes the elements of each $\mathcal{B}(h)$.

Let

$$h^\star = (\mathcal{O}_1, \mathcal{O}_{t+1}) \cdots (\mathcal{O}_t, \mathcal{O}_{2t}) \in S_{\{\mathcal{O}_1, \ldots, \mathcal{O}_r\}}$$

and let $s_{\omega^\star}\overline{\omega^\star} \in \mathcal{B}(h^\star)$ be the unique basis element such that

$$\mathcal{I}(\omega^\star) = (1, tp+1)(2, tp+2) \cdots (tp, 2tp)$$

(Equivalently, $\mathcal{I}(\omega^\star)$ is the unique involution in S_{2tp} that preserves the relative orders of the elements in $\mathcal{O}_j$ for $1 \le j \le 2t$ and satisfies $\mathcal{I}_{\mathcal{O}}(\omega^\star) = h^\star$.) By Lemma 3.3(ii) we see that the stabiliser of $\overline{\omega^\star}$ in C is the subgroup E_t. Let $s_\delta\overline{\delta} \in B(h^\star)$, then $\mathcal{I}(\delta) = \mathcal{I}(\omega^\star) = h^\star$. Without loss of generality we have that

$$\mathcal{I}(\delta) = (1, i_1)(2, i_2) \cdots (tp, i_{tp}),$$

where $\{i_{(j-1)p+1}, i_{(j-1)p+2}, \ldots, i_{jp}\} = \mathcal{O}_{t+j}$ for all $j \in \{1, \ldots, t\}$. Hence, there exist $k_1, k_2, \ldots, k_t \in \{0, 1, \ldots, p-1\}$ and a permutation

$$g = z_{t+1}^{k_1} z_{t+2}^{k_2} \cdots z_{2t}^{k_t},$$

such that $(tp + (j-1)p + 1)g = i_{(j-1)p+1}$ for all $j \in \{1, \ldots, t\}$. Since $s_\delta\overline{\delta}$ is fixed by R_r, it follows that $s_{\omega^\star}\overline{\omega^\star}g = s_\delta\overline{\delta}$. Therefore any basis element in $\mathcal{B}(h^\star)$ can be obtained from $\omega^\star$ by permuting the members of $\mathcal{O}_{t+1}, \ldots, \mathcal{O}_{2t}$ by an element of C. It follows that $\mathcal{B}(h^\star)$ has size p^t and is equal to the orbit of $s_{\omega^\star}\overline{\omega^\star}$ on C. Therefore there is an isomorphism of FC-modules $\langle \mathcal{B}(h^\star) \rangle \cong F{\uparrow}_{E_t}^{C}$. By Lemma 4.1, $\langle \mathcal{B}(h^\star) \rangle$ is an indecomposable FC-module with vertex E_t.

For each involution $h \in S_{\{\mathcal{O}_1,\ldots,\mathcal{O}_r\}}$, the FC-submodule $\langle \mathcal{B}(h) \rangle$ of M is sent to $\langle \mathcal{B}(h^\star) \rangle$ by an element of $N_{S_{rp}}(R_r)$ normalizing C. It follows that if U is any summand of M, now considered as an $FN_{S_{rp}}(R_r)$-module, then the restriction of U to C is isomorphic to a direct sum of indecomposable p-permutation FC-modules with vertices conjugate in $N_{S_{rp}}(R_r)$ to E_t. Applying Theorem 2.2 to these summands, we see that there exists $g \in N_{S_{rp}}(R_r)$ such that $U(E_t^g) \neq 0$. Now by Theorem 2.2, this time applied to the $FN_{S_{rp}}(R_r)$-module U, we see that U has a vertex containing E_t^g. Hence every indecomposable summand of M has a vertex containing E_t.

We now calculate the Brauer correspondent $M(E_t)$. Let $s_\omega\overline{\omega} \in \mathcal{B}$. It follows from Lemma 3.3(ii) that $\overline{\omega}$ is fixed by E_t if and only if $\mathcal{I}_{\mathcal{O}}(\omega)$ is the involution $h^\star$. Hence, by Corollary 2.3 and Lemma 3.3, we have $M(E_t) = \langle \mathcal{B}(h^\star) \rangle$. We have already seen that $\langle \mathcal{B}(h^\star) \rangle$ is indecomposable as an FC-module. Since C normalizes E_t and centralizes R_r, it follows that $M(E_t)$ is indecomposable as a module for the normalizer of E_t in $N_{S_{rp}}(R_r)$. We already know that every indecomposable summand of M has a vertex containing E_t, so it follows from Corollary 2.3 that M is indecomposable. □

Note that if $\omega^\star$ is as defined in the proof of Proposition 4.4, then Q_t is a Sylow p-subgroup of $\mathrm{C}_{S_{rp}}\big(\mathcal{I}(\omega^\star)\big) \cong (S_2 \wr S_{tp}) \times S_{(r-2t)p}$. Using this observation and the p-permutation basis $\mathcal{B}$ for M it is now straightforward to prove the following proposition.

Proposition 4.5. *The indecomposable $FN_{S_{rp}}(R_r)$-module M has Q_t as a vertex.*

Proof. By Corollary 2.3, if Q is subgroup of P maximal subject to $\mathcal{B}^Q \neq \varnothing$ then Q is a vertex of M. By Lemma 3.3(ii), a basis element $s_\omega \overline{\omega} \in B$ is fixed by a p-subgroup Q of P if and only if $Q \leq \mathrm{C}_{S_{rp}}\big(\mathcal{I}(\omega)\big)$. Taking $\omega = \omega^\star$ we see that there is a vertex of M containing Q_t. On the other hand, $\mathrm{C}_{S_{rp}}\big(\mathcal{I}(\omega)\big)$ is conjugate in S_{rp} to $\mathrm{C}_{S_{rp}}\big(\mathcal{I}(\omega^\star)\big)$, and so if $Q \leq \mathrm{C}_{S_{rp}}\big(\mathcal{I}(\omega)\big)$ then $|Q| \leq |Q_t|$. It follows that Q_t is a vertex of M. □

Third step: proof of Theorem 1.2. For the remainder of the proof we shall regard $S_{(r-2t)p}$ as acting on $\{2tp+1, \ldots, rp\}$. We denote by D_t the p-group $C \cap N_{S_{rp}}(Q_t)$. Notice that $\langle D_t, Q_t\rangle$ is a p-group since is a subgroup of $\langle C, Q_t\rangle \leq P$. We shall need the following lemma to work with modules for $N_{S_{rp}}(Q_t)$.

Lemma 4.6. *The unique Sylow p-subgroup of $N_{S_{rp}}(Q_t)$ is the subgroup $\langle D_t, Q_t\rangle$ of P.*

Proof. Let $x \in N_{S_{rp}}(Q_t)$. If $2t+1 \leq j \leq r$ then the conjugate z_j^x of the p-cycle $z_j \in E_t$ is a p-cycle in Q_t. Since Q_t normalizes E_t, it permutes the orbits $\mathcal{O}_1, \ldots, \mathcal{O}_r$ of E_t as blocks for its action. No p-cycle can act non-trivially on these blocks, so $z_j^x \in \langle z_{2t+1}, \ldots, z_r\rangle$. Hence if $1 \leq j \leq t$ then $(z_j z_{j+t})^x \in \langle z_1 z_{t+1}, \ldots, z_t z_{2t}\rangle$. It follows that $N_{S_{rp}}(Q_t)$ factorizes as

$$N_{S_{rp}}(Q_t) = N_{S_{2tp}}(L) \times N_{S_{(r-2t)p}}(L^+)$$

where L and L^+ are as defined at the start of the second step. Moreover, we see that $N_{S_{rp}}(Q_t)$ permutes, as blocks for its action, the sets $\mathcal{O}_1 \cup \mathcal{O}_{t+1}, \ldots, \mathcal{O}_t \cup \mathcal{O}_{2t}$ and $\mathcal{O}_{2t+1}, \ldots, \mathcal{O}_r$.

Let $h \in N_{S_{rp}}(Q_t)$ be a p-element. We may factorize h as gg^+ where $g \in N_{S_{2tp}}(L)$ and $g^+ \in N_{S_{(r-2t)p}}(L^+)$ are p-elements. Since $\langle L^+, g^+\rangle$ is a p-group and L^+ is a Sylow p-subgroup of $S_{(r-2t)p}$, we have $g^+ \in L^+$. Let

$$X = \{\mathcal{O}_1 \cup \mathcal{O}_{t+1}, \ldots, \mathcal{O}_t \cup \mathcal{O}_{2t}\}.$$

The group $\langle L, g\rangle$ permutes the sets in X as blocks for its action. Let

$$\pi : \langle L, g\rangle \to S_X$$

be the corresponding group homomorphism. By construction L acts on the sets $\mathcal{O}_1, \ldots, \mathcal{O}_t$ as a Sylow p-subgroup of $S_{\{\mathcal{O}_1, \ldots, \mathcal{O}_t\}}$; hence $L\pi$ is a Sylow p-subgroup of S_X. Since $\langle L, g\rangle$ is a p-group, there exists $\tilde{g} \in L$ such that $g\pi = \tilde{g}\pi$. Let $y = g\tilde{g}^{-1}$. Since y acts trivially on X, we may write

$$y = g_1 \ldots g_t$$

where $g_j \in S_{\mathcal{O}_j \cup \mathcal{O}_{j+t}}$ for each j. The p-group $\langle L, y\rangle$ has as a subgroup $\langle z_j z_{j+t}, y\rangle$. The permutation group induced by this subgroup on $\mathcal{O}_j \cup \mathcal{O}_{j+t}$, namely $\langle z_j z_{j+t}, g_j\rangle$, is a p-group acting on a set of size $2p$. Since p is odd, the unique Sylow p-subgroup of $S_{\mathcal{O}_j \cup \mathcal{O}_{j+t}}$ containing $z_j z_{j+t}$ is $\langle z_j, z_{j+t}\rangle$. Hence

$g_j \in \langle z_j, z_{j+t}\rangle$ for each j. Therefore $y \in \langle z_1, \ldots, z_t, z_{t+1}, \ldots, z_{2t}\rangle \leq C$. We also know that $y \in \langle Q_t, g\rangle \leq N_{S_{2tp}}(Q_t) \leq N_{S_{rp}}(Q_t)$. Therefore $y \in D_t$, and since $\tilde{g} \in Q_t$, it follows that $g \in \langle D_t, Q_t\rangle$. Hence $h = gg^+ \in \langle D_t, Q_t\rangle \leq \langle C, Q_t\rangle \leq P$.

Conversely, the subgroup $\langle D_t, Q_t\rangle$ is contained in $N_{S_{rp}}(Q_t)$ because both D_t and Q_t are. It follows that $\langle D_t, Q_t\rangle$ is the unique Sylow p-subgroup of $N_{S_{rp}}(Q_t)$. □

We also need the following two general lemmas.

Lemma 4.7. *Let Q and R be p-subgroups of a finite group G and let U be a p-permutation FG-module. Let $K = N_G(R)$. If R is normal in Q then the Brauer correspondents $U(Q)$ and $\big(U(R)\big)(Q)$ are isomorphic as $FN_K(Q)$-modules.*

Proof. Let P be a Sylow p-subgroup of $N_G(R)$ containing Q and let $\mathcal{B}$ be a p-permutation basis for U with respect to a Sylow p-subgroup of G containing P. By Corollary 2.3 we have $U(Q) = \langle \mathcal{B}^Q\rangle$ as an $FN_G(Q)$-module. In particular

$$U(Q)\!\downarrow_{N_K(Q)} = \langle \mathcal{B}^Q\rangle$$

as an $FN_K(Q)$-module. On the other hand $U(R) = \langle \mathcal{B}^R\rangle$ as an $FN_G(R)$-module. Now $\mathcal{B}^R$ is a p-permutation basis for $U(R)$ with respect to $K \cap P$. Since this subgroup contains Q we have $\big(U(R)\big)(Q) = \langle \mathcal{B}^R\rangle(Q) = \langle (\mathcal{B}^R)^Q\rangle = \langle \mathcal{B}^Q\rangle$ as $FN_K(Q)$-modules, as required. □

Lemma 4.8. *Let G and G' be finite groups and let U and U' be p-permutation modules for FG and FG', respectively. If $Q \leq G$ is a p-subgroup then $(U \boxtimes U')(Q) = U(Q) \boxtimes U'$, where on the left-hand side Q is regarded as a subgroup of $G \times G'$ in the obvious way.*

Proof. This follows from Corollary 2.3 by taking p-permutation bases for U and U' such that the p-permutation basis for U is permuted by a Sylow p-subgroup of G containing Q. □

We are now ready to prove Theorem 1.2. We repeat the statement below for the reader's convenience.

Theorem 1.2. *Let $m \in \mathbf{N}$ and let $k \in \mathbf{N}_0$. If U is an indecomposable non-projective summand of $H^{(2^m\,;\,k)}$, defined over a field F of odd characteristic p, then U has as a vertex a Sylow p-subgroup Q of $(S_2 \wr S_{tp}) \times S_{(r-2t)p}$ for some $t \in \mathbf{N}_0$ and $r \in \mathbf{N}$ with $tp \leq m$, $2t \leq r$ and $(r-2t)p \leq k$. Moreover the Green correspondent of U admits a tensor factorization $V \boxtimes W$ as a module for $F\big((N_{S_{rp}}(Q)/Q) \times S_{2m+k-rp}\big)$, where V and W are projective, and W is an indecomposable summand of $H^{(2^{m-tp}\,;\,k-(r-2t)p)}$.*

Proof. Let $r \in \mathbf{N}$ be maximal such that the subgroup R_r is contained in a vertex of U. Let $K = N_{S_{rp}}(R_r)$. By Lemma 4.2 and Proposition 4.3 there is an isomorphism of $N_{S_{2m+k}}(R_r)$-modules

$$H^{(2^m\,;\,k)}(R_r) \cong \bigoplus_{t\in T_r} \bigl(H^{(2^{tp}\,;\,(r-2t)p)}(R_r) \boxtimes H^{(2^{m-tp}\,;\,k-(r-2t)p)}\bigr)$$

compatible with the factorization $N_{S_{2m+k}}(R_r) = K \times S_{2m+k-rp}$, where we regard $S_{2m+k-rp}$ as acting on $\{rp+1,\ldots,2m+k\}$ and shift each module $H^{(2^{m-tp}\,;\,k-(r-2t)p)}$ appropriately.

For $t \in T_r$, let $M_t = H^{(2^{tp}\,;\,(r-2t)p)}(R_r)$. By Proposition 4.5, each M_t is indecomposable as an $FN_{S_{rp}}(R_r)$-module. Hence, by the Krull–Schmidt Theorem, there is a subset $T' \subset T_r$, and for each $t \in T'$, a non-zero summand W_t of $H^{(2^{m-tp}\,;\,k-(r-2t)p)}$, such that

$$U(R_r) \cong \bigoplus_{t\in T'} M_t \boxtimes W_t$$

as $F(K \times S_{2m+k-rp})$-modules. By Proposition 4.5, M_t has Q_t as a vertex for each non-zero $t \in T'$. It is clear that $M_0 = \mathrm{sgn}_{S_{rp}}(R_r)$ has vertex Q_0 as an $FN_{S_{rp}}(R_r)$-module. Let ℓ be the least element of T'. If $t > \ell$ then Q_t does not contain a conjugate of the subgroup E_ℓ of Q_ℓ. Hence, by Theorem 2.2, we have $M_t(Q_\ell) = 0$. It now follows from Lemmas 4.7 and 4.8 that there is an isomorphism of $F(N_K(Q_\ell) \times S_{2m+k-rp})$-modules

$$U(Q_\ell) \cong U(R_r)(Q_\ell) \cong M_\ell(Q_\ell) \boxtimes W_\ell.$$

Since M_ℓ has Q_ℓ as a vertex, we have $M_\ell(Q_\ell) \neq 0$. It follows that U has a vertex Q containing Q_ℓ.

Let $\mathcal{B}$ be the p-permutation basis for M_ℓ defined in the second step. Since $\mathcal{B}$ is permuted by the Sylow p-subgroup P of K, it follows from Corollary 2.3 and Lemma 4.6 that $\mathcal{C} = \mathcal{B}^{Q_\ell}$ is a p-permutation basis for the $FN_K(Q_\ell)$-module $M_\ell(Q_\ell)$ with respect to the Sylow p-subgroup $\langle D_\ell, Q_\ell\rangle$ of $N_K(Q_\ell)$. Since W_ℓ is isomorphic to a direct summand of the p-permutation module $H^{(2^{m-\ell p}\,;\,k-(r-2\ell)p)}$ it has a p-permutation basis $\mathcal{C}^+$ with respect to a Sylow p-subgroup P^+ of $S_{\{rp+1,\ldots,2m+k\}}$. Therefore

$$\mathcal{C} \boxtimes \mathcal{C}^+ = \{v \otimes v^+ : v \in \mathcal{C}, v^+ \in \mathcal{C}^+\}$$

is a p-permutation basis for $M_\ell(Q_\ell) \boxtimes W_\ell$ with respect to the Sylow subgroup $\langle D_\ell, Q_\ell\rangle \times P^+$ of $N_K(Q_\ell) \times S_{2m+k-rp}$.

Suppose, for a contradiction, that Q strictly contains Q_ℓ. Since Q is a p-group there exists a p-element $g \in N_Q(Q_\ell) \leq N_{S_{2m+k}}(Q_\ell)$ such that $g \notin Q_\ell$. Now Q_ℓ has orbits of length at least p on $\{1,\ldots,rp\}$ and fixes $\{rp+1,\ldots,2m+k\}$. Since g permutes these orbits as blocks for its action, we may factorize g as $g = hh^+$ where $h \in N_{S_{rp}}(Q_\ell)$ and $h^+ \in S_{2m+k-rp}$. By Lemma 4.6 we have that $\langle Q_\ell, h\rangle \leq N_K(Q_\ell)$.

Corollary 2.3 now implies that $(\mathcal{C} \boxtimes \mathcal{C}^+)^{\langle Q_\ell, g\rangle} \neq \varnothing$. Let $v \otimes v^+ \in \mathcal{C} \boxtimes \mathcal{C}^+$ be such that $(v \otimes v^+)g = v \otimes v^+$. Then $v \in \mathcal{B}^{\langle Q_\ell, h\rangle}$. But Q_ℓ is a vertex of M_ℓ, so it follows from Corollary 2.3 that $h \in Q_\ell$. Hence h^+ is a non-identity element of Q. By taking an appropriate power of h^+ we find that Q contains a product of one or more p-cycles with support contained in $\{rp+1, \ldots, 2m+k\}$. This contradicts our assumption that r was maximal such that R_r is contained in a vertex of U.

Therefore U has vertex Q_ℓ. We saw above that there is an isomorphism $U(Q_\ell) \cong M_\ell(Q_\ell) \boxtimes W_\ell$ of $F(N_K(Q_\ell) \times S_{2m+k-rp})$-modules. This identifies $U(Q_\ell)$ as a vector space on which $N_{S_{2m+k}}(Q_\ell) = N_{S_{rp}}(Q_\ell) \times S_{2m+k-rp}$ acts. It is clear from the isomorphism in Proposition 4.3 that $N_{S_{rp}}(Q_\ell)$ acts on the first tensor factor and $S_{2m+k-rp}$ acts on the second. Hence the action of $N_K(Q_\ell)$ on $M_\ell(Q_\ell)$ extends to an action of $N_{S_{rp}}(Q_\ell)$ on $M_\ell(Q_\ell)$ and we obtain a tensor factorization $V \boxtimes W_\ell$ of $U(Q_\ell)$ as an $N_{S_{rp}}(Q_\ell) \times S_{2m+k-rp}$-module. An outer tensor product of modules is projective if and only if both factors are projective, so by Theorem 2.5, V is a projective $FN_{S_{rp}}(Q_\ell)/Q_\ell$-module, W_ℓ is a projective $FS_{2m+k-rp}$-module, and $U(Q_\ell)$ is the Green correspondent of U. □

5. Proofs of Theorem 1.1 and Proposition 1.3

In this section we prove Proposition 1.3, and hence Theorem 1.1. It will be convenient to assume that $H^{(2^m\,;\,k)}$ is defined over the finite field $\mathbf{F}_p$. Proposition 1.3 then follows for an arbitrary field of characteristic p by change of scalars. We assume the common hypotheses for these results, so γ is a p-core such that $2m+k = |\gamma| + w_k(\gamma)p$ and if $k \geq p$ then

$$w_{k-p}(\gamma) \neq w_k(\gamma) - 1.$$

Let λ be a maximal element of $\mathcal{E}_k(\gamma)$ under the dominance order.

Write $H_{\mathbf{Q}}^{(2^m\,;\,k)}$ for the twisted Foulkes module defined over the rational field. This module has an ordinary character given by Lemma 3.1. In particular it has χ^λ as a constituent, and so the rational Specht module $S_{\mathbf{Q}}^\lambda$ is a direct summand of $H_{\mathbf{Q}}^{(2^m\,;\,k)}$. Therefore, by reduction modulo p, each composition factor of S^λ (now defined over $\mathbf{F}_p$) appears in $H^{(2^m\,;\,k)}$. In particular $H^{(2^m\,;\,k)}$ has a summand in the block $B(\gamma, w_k(\gamma))$ with p-core γ and weight $w_k(\gamma)$. We now use Theorem 1.2 to show that any such summand is projective.

Proposition 5.1. *Every summand of $H^{(2^m\,;\,k)}$ in the block $B(\gamma, w_k(\gamma))$ of S_{2m+k} is projective.*

Proof. Suppose, for a contradiction, that $H^{(2^m\,;\,k)}$ has a non-projective summand U in $B(\gamma, w_k(\gamma))$. By Theorem 1.2, the vertex of U is a Sylow subgroup Q_t of $(S_2 \wr S_{tp}) \times S_{(r-2t)p}$ for some $r \in \mathbf{N}$ and $t \in \mathbf{N}_0$ such that $tp \leq m$, $2t \leq r$ and $(r-2t)p \leq k$.

Suppose first of all that $2t < r$. In this case there is a p-cycle $g \in Q_t$. Replacing Q_t with a conjugate, we may assume that $g = (1, \ldots, p)$ and so $\langle g \rangle = R_1$ where R_1 is as defined at the start of the first step in §4. By Lemma 4.2 and Proposition 4.3, we have that $k \geq p$ and $U(R_1)$ is a direct summand of

$$H^{(2^m\,;\,k)}(R_1) = \mathrm{sgn}_{S_p}(\langle g \rangle) \boxtimes H^{(2^m\,;\,k-p)}.$$

Hence there exists an indecomposable summand W of $H^{(2^m\,;\,k-p)}$ such that

$$\mathrm{sgn}_{S_p}(\langle g \rangle) \boxtimes W \mid U(R_1).$$

By Theorem 2.7, W lies in the block $B(\gamma, w_k(\gamma) - 1)$ of S_{2m+k-p}. In particular, this implies that $H^{(2^m\,;\,k-p)}$ has a composition factor in this block. Therefore there is a constituent χ^μ of the ordinary character of $H^{(2^m\,;\,k-p)}$ such that S^μ lies in $B(\gamma, w_k(\gamma) - 1)$. But then, by Lemma 3.1, μ is a partition with p-core γ having exactly $k - p$ odd parts and weight $w_k(\gamma) - 1$. Adding a single vertical p-hook to μ gives a partition of weight $w_k(\gamma)$ with exactly k odd parts. Hence $w_{k-p}(\gamma) = w_k(\gamma) - 1$, contrary to the hypothesis on $w_{k-p}(\gamma)$.

Now suppose that $2t = r$. Let $g = (1, \ldots, p)(p+1, \ldots, 2p)$. Then $g \in Q_t$ by definition and $\langle g \rangle = R_2$. By Lemma 4.2 and Proposition 4.3 we have that $U(R_2)$ is a direct summand of

$$H^{(2^m\,;\,k)}(R_2) = \Big(H^{(2^p)}(\langle g \rangle) \boxtimes H^{(2^{m-p}\,;\,k)}\Big) \bigoplus \Big(\mathrm{sgn}_{S_{2p}}(\langle g \rangle) \boxtimes H^{(2^m\,;\,k-2p)}\Big)$$

where the second summand should be disregarded if $k < 2p$. It follows that either there is an indecomposable $FS_{2m+k-2p}$-module V such that

$$H^{(2^p)}(\langle g \rangle) \boxtimes V \mid U(R_2),$$

or $k \geq 2p$ and there is an indecomposable $FS_{2m+k-2p}$-module W such that

$$\mathrm{sgn}_{S_{2k}}(\langle g \rangle) \boxtimes W \mid U(R_2).$$

Again we use Theorem 2.7. In the first case V lies in the block $B(\gamma, w_k(\gamma) - 2)$ of $S_{2m+k-2p}$. Hence there is a constituent χ^μ of the ordinary character of $H^{(2^{m-p}\,;\,k)}$ such that μ is a partition with p-core γ and weight $w_k(\gamma) - 2$ having exactly k odd parts. This contradicts the minimality of $w_k(\gamma)$. In the second case W also lies in the block $B(\gamma, w_k(\gamma) - 2)$ of $S_{2m+k-2p}$ and there is a constituent χ^μ of the ordinary character of $H^{(2^m\,;\,k-2p)}$ such that μ is a partition with p-core γ and weight $w_k(\gamma) - 2$ having exactly $k - 2p$ odd parts. But then by adding a single vertical p-hook to μ we reach a partition with weight $w_k(\gamma)-1$ having exactly $k-p$ odd parts. Once again this contradicts the hypothesis that $w_{k-p}(\gamma) \neq w_k(\gamma)-1$. $\square$

For ν a p-regular partition, let P^ν denote the projective cover of the simple module D^ν. To finish the proof of Proposition 1.3 we must show that if λ is a maximal element of $\mathcal{E}_k(\gamma)$ then P^λ is one of the projective summands of $H^{(2^m\,;\,k)}$ in the block $B(\gamma, w_k(\gamma))$. For this we need a lifting result for summands of the monomial module $H^{(2^m\,;\,k)}$, which we prove using the analogous, and well known, result for permutation modules. Let $\mathbf{Z}_p$ denote the ring of p-adic integers and let $H^{(2^m\,;\,k)}_{\mathbf{Z}_p}$ denote the twisted Foulkes module defined over $\mathbf{Z}_p$.

Lemma 5.2. *If U is a direct summand of $H^{(2^m\,;\,k)}$ then there is a $\mathbf{Z}_pS_{2n+k}$-module $U_{\mathbf{Z}_p}$, unique up to isomorphism, such that $U_{\mathbf{Z}_p}$ is a direct summand of $H^{(2^m\,;\,k)}_{\mathbf{Z}_p}$ and $U_{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{F}_p \cong U$.*

Proof. Let A_k denote the alternating group on $\{2m+1,\ldots,2m+k\}$. Let $M = \mathbf{F}_p\big\uparrow_{S_2 \wr S_m \times A_k}^{S_{2m+k}}$ be the permutation module of S_{2m+k} acting on the cosets of $S_2 \wr S_m \times A_k$ and let $M_{\mathbf{Z}_p} = \mathbf{Z}_p\big\uparrow_{S_2 \wr S_m \times A_k}^{S_{2m+k}}$ be the corresponding permutation module defined over $\mathbf{Z}_p$. Since p is odd, the trivial $\mathbf{Z}_p(S_2 \wr S_m \times S_k)$ module is a direct summand of $\mathbf{Z}_p\big\uparrow_{S_2 \wr S_m \times A_k}^{S_2 \wr S_m \times S_k}$. Hence, inducing up to S_{2m+k} (as in the remark after Lemma 3.2), we see that $M_{\mathbf{Z}_p} = H^{(2^m\,;\,k)}_{\mathbf{Z}_p} \oplus M'_{\mathbf{Z}_p}$ where $M'_{\mathbf{Z}_p}$ is a complementary $\mathbf{Z}_pS_{2m+k}$-module, and $M = H^{(2^m\,;\,k)} \oplus M'$ where M' is the reduction modulo p of $M'_{\mathbf{Z}_p}$.

By Scott's lifting theorem (see [2, Theorem 3.11.3]), reduction modulo p is a bijection between the summands of $M_{\mathbf{Z}_p}$ and the summands of M. By the same result, this bijection restricts to a bijection between the summands of the permutation module $M'_{\mathbf{Z}_p}$ and the summands of M'. Since U is a direct summand of M there is a summand $U_{\mathbf{Z}_p}$ of $M_{\mathbf{Z}_p}$, unique up to isomorphism, such that $U_{\mathbf{Z}_p} \otimes_{\mathbf{Z}_p} \mathbf{F}_p \cong U$. By the remarks just made, $U_{\mathbf{Z}_p}$ is isomorphic to a summand of $H^{(2^m\,;\,k)}_{\mathbf{Z}_p}$. $\square$

Let $P^\nu_{\mathbf{Z}_p}$ be the $\mathbf{Z}_p$-free $\mathbf{Z}_pS_{2m+k}$-module whose reduction modulo p is P^ν. By Brauer reciprocity (see for instance [39, §15.4]), the ordinary character of $P^\nu_{\mathbf{Z}_p}$ is

$$(\star) \qquad \psi^\nu = \sum_\mu d_{\mu\nu}\chi^\mu.$$

The result mentioned in the introduction, that if $d_{\mu\nu} \neq 0$ then ν dominates μ, implies that the sum may be taken over those partitions μ dominated by ν.

Proof of Proposition 1.3. We have seen that each summand of $H^{(2^m\,;\,k)}$ in the block $B(\gamma, w_k(\gamma))$ is projective and that there is at least one such summand. Let $P^{\nu_1}, \ldots, P^{\nu_c}$ be the summands of $H^{(2^m\,;\,k)}$ in $B(\gamma, w_k(\gamma))$. Using

Lemma 5.2 to lift these summands to summands of $H^{(2^m\,;\,k)}_{\mathbf{Z}_p}$ we see that the ordinary character of the summand of $H^{(2^m\,;\,k)}_{\mathbf{Z}_p}$ lying in the p-block of S_{2m+k} with core γ and weight $w_k(\gamma)$ is $\psi^{\nu_1} + \cdots + \psi^{\nu_c}$. By Lemma 3.1 we have

$$(\dagger) \qquad \psi^{\nu_1} + \cdots + \psi^{\nu_c} = \sum_{\mu \in \mathcal{E}_k(\gamma)} \chi^{\mu}.$$

By hypothesis λ is a maximal partition in the dominance order on $\mathcal{E}_k(\gamma)$, and by $(\star)$ each ψ^{ν_j} is a sum of ordinary irreducible characters χ^{μ} for partitions μ dominated by ν_j. Therefore one of the partitions ν_j must equal λ, as required. □

We are now ready to prove Theorem 1.1

Proof of Theorem 1.1. Suppose that the projective summands of $H^{(2^m\,;\,k)}$ lying in the block $B(\gamma, w_k(\gamma))$ are $P^{\nu_1}, \ldots, P^{\nu_c}$. Then by $(\dagger)$ above, $\mathcal{E}_k(\gamma)$ has a partition into disjoint subsets $\mathcal{X}_1, \ldots, \mathcal{X}_c$ such that $\nu_j \in \mathcal{X}_j$ and

$$\psi^{\nu_j} = \sum_{\mu \in \mathcal{X}_j} \chi^{\mu}$$

for each j. It now follows from $(\star)$ that the column of the decomposition matrix of S_n in characteristic p labelled by ν_j has 1s in the rows labelled by partitions in $\mathcal{X}_j$, and 0s in all other rows. □

6. Applications of Theorem 1.1 and Proposition 1.3

We begin with a precise statement of the result on diagonal Cartan numbers mentioned in the introduction after Proposition 1.3.

Theorem 6.1 ([37, Theorem 2.8] or [3, Proposition 4.6(i)])**.** *If ν is a p-regular partition of n such that ν has weight w then $d_{\mu\nu} \neq 0$ for at least $w+1$ distinct partitions μ.*

If $|\mathcal{E}_k(\gamma)| \leq 2w_k(\gamma) + 1$ then it follows from Theorems 1.1 and 6.1 that $\mathcal{E}_k(\gamma)$ has a unique maximal partition, say λ, and the only non-zero entries of the column of the decomposition matrix of S_n labelled by λ are 1s in rows labelled by partitions in $\mathcal{E}_k(\gamma)$. In these cases Theorem 1.1 becomes a sharp result.

Example 6.2. Firstly let $p = 3$ and let $\gamma = (3,1,1)$. We leave it to the reader to check that $w_0(\gamma) = 3$ and

$$\mathcal{E}_0(\gamma) = \{(8,4,2), (6,6,2), (6,4,4), (6,4,2,2)\}.$$

Hence the column of the decomposition matrix of S_{12} in characteristic 3 labelled by $(8,4,2)$ has 1s in the rows labelled by the four partitions in $\mathcal{E}_0(\gamma)$ and no other non-zero entries. (The full decomposition matrix of the block $B((3,1,1),3)$ was found by Fayers in [11, A.8].)

Secondly let $p = 7$ and let $\gamma = (4, 4, 4)$. Then $w_6(\gamma) = 2$ and $\mathcal{E}_6(\gamma) = \mathcal{X} \cup \mathcal{X}'$ where

$$\mathcal{X} = \{(11, 4, 4, 3, 1^4), (11, 4, 4, 2, 1^5), (10, 5, 4, 3, 1^4), (10, 5, 4, 2, 1^5)\},$$
$$\mathcal{X}' = \{(9, 5, 5, 5, 1, 1), (9, 5, 5, 4, 1, 1, 1), (8, 5, 5, 5, 1, 1, 1)\}.$$

The partitions in $\mathcal{X}$ and $\mathcal{X}'$ are mutually incomparable under the dominance order. Thus Theorem 1.1 determines the columns of the decomposition matrix of S_{26} in characteristic 7 labelled by $(11, 4, 4, 3, 1^4)$ and $(9, 5, 5, 5, 1, 1)$.

Finally let $p = 5$ and let $\gamma = (5, 4, 2, 1^4)$. Then $w_6(\gamma) = 3$, and

$$\mathcal{E}_6(\gamma) = \left\{ \begin{array}{c} (15, 9, 2, 1^4), (15, 6, 5, 1^4), (13, 11, 2, 1^4) \\ (13, 6, 5, 3, 1^3), (10, 9, 7, 1^4), (10, 9, 5, 3, 1^3) \end{array} \right\}.$$

It is easily seen that $w_1(\gamma) > 2$. (In fact $w_1(\gamma) = 8$.) Therefore Theorem 1.1 determines the column of the decomposition matrix of S_{30} in characteristic 5 labelled by $(15, 9, 2, 1^4)$.

We now use the following combinatorial lemma to prove that the bound in Theorem 6.1 is attained in blocks of every weight. Note that when $p = 3$ and $e = 2$ the core used is $(3, 1, 1)$, as in the first example above. For an introduction to James' abacus see [21, page 78].

Lemma 6.3. *Let p be an odd number, let $e \in \mathbf{N}_0$, and let γ be the p-core represented by the p-abacus with two beads on runner 1, $e + 1$ beads on runner $p - 1$, and one bead on every other runner. If $0 \leq k \leq e + 1$ then $w_k(\gamma) = e + 1 - k$ and $|\mathcal{E}_k(\gamma)| = w_k(\gamma) + 1$.*

Proof. The p-core γ is represented by the abacus A shown in Figure 1 overleaf. Moving the lowest $e+1-k$ beads on runner $p-1$ down one step leaves a partition with exactly k odd parts. Therefore $w_k(\gamma) \leq e + 1 - k$.

Suppose that λ is a partition with exactly k odd parts that can be obtained by a sequence of single step bead moves on A in which exactly $e - r$ beads are moved on runner $p - 1$ and at most $e + 1 - k$ moves are made in total. We may suppose that $r \geq k$ and that the beads on runner $p - 1$ are moved first, leaving an abacus $A^\star$. Numbering rows as in Figure 1, so that row 0 is the highest row, let row ℓ be the lowest row of $A^\star$ to which any bead is moved in the subsequent moves. Let B be the abacus representing λ that is obtained from $A^\star$ by making these moves. The number of spaces before each beads on runner $p-1$ in rows ℓ, $\ell+1$, ..., r is the same in both $A^\star$ and B, and is clearly odd in $A^\star$. Hence the parts corresponding to these beads are odd. Therefore $\ell \geq r - k + 1$.

If B has a bead in row ℓ on a runner other than runner 1 or runner $p-1$, then this bead has been moved down from row 0, and so has been moved at least ℓ times. The total number of moves made is at least $(e-r)+\ell \geq e-k+1$,

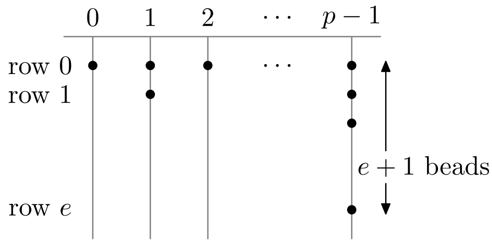

FIGURE 1. Abacus A representing the p-core γ in Lemma 6.3.

and so $\ell = r - k + 1$. But now B has beads corresponding to odd parts of λ on runner $p-1$ in row 0, as well as rows ℓ, $\ell+1$, $\ldots$, r, giving $k+1$ odd parts in total, a contradiction.

It follows that the sequence of bead moves leading to B may be reordered so that the first $e-r$ moves are made on runner $p-1$, and then the lowest bead on runner 1 is pushed down $r-k$ times to row $r-k+1$. The partition after these moves has $k+1$ odd parts. Moving the bead on runner 1 down one step from row $r-k+1$ reduces the number of odd parts by one, and is the only such move that does not move a bead on runner $p-1$. Therefore $\mathcal{E}_k(\gamma)$ contains the partition constructed at the start of the proof, and one further partition for each $r \in \{0, 1, \ldots, e-k\}$. $\square$

Given an arbitrary weight $w \in \mathbf{N}$ and $k \in \mathbf{N}_0$, Lemma 6.3 gives an explicit partition λ satisfying the hypothesis of Theorem 1.1 and such that $w_k(\gamma) = w$. We use this in the following proposition.

Proposition 6.4. *Let p be an odd prime and let k, $w \in \mathbf{N}_0$ be given. There exists a p-core γ and a partition λ with p-core γ and weight w such that λ has exactly k odd parts and the only non-zero entries in the column of the decomposition matrix labelled by λ are 1s lying in the $w+1$ rows labelled by elements of $\mathcal{E}_k(\gamma)$.*

Proof. If $w = 0$ and $k = 0$ then take $\lambda = (2)$. Otherwise let γ be the p-core in Lemma 6.3 when $e = w + k - 1$. By this lemma we have $w_k(\gamma) = w$. Moreover, if $k \geq p$ then $w_{k-p}(\gamma) = w + p$. Taking λ to be a maximal element of $\mathcal{E}_k(\gamma)$, the proposition follows from Theorem 1.1 and Theorem 6.1. $\square$

We now turn to an application of Proposition 1.3. Write $H_R^{(2^m\,;\,k)}$ for the twisted Foulkes module defined over a commutative ring R. Since the ordinary character of $H_{\mathbf{Q}}^{(2^m\,;\,k)}$ is multiplicity-free, the endomorphism algebra of $H_F^{(2^m\,;\,k)}$ is commutative whenever the field F has characteristic zero. Hence the endomorphism ring of $H_{\mathbf{Z}}^{(2^m\,;\,k)}$ is commutative. This ring has a canonical $\mathbf{Z}$-basis indexed by the double cosets of the subgroup $S_2 \wr S_m \times S_k$

in S_{2m+k}. This basis makes it clear that the canonical map

$$\mathrm{End}_{\mathbf{Z}S_{2m+k}}(H_{\mathbf{Z}}^{(2^m\,;\,k)}) \to \mathrm{End}_{\mathbf{F}_pS_{2m+k}}(H_{\mathbf{F}_p}^{(2^m\,;\,k)})$$

is surjective, and so $\mathrm{End}_{FS_{2m+k}}(H_F^{(2^m\,;\,k)})$ is commutative for *any* field F. This fact has some strong consequences for the structure of twisted Foulkes modules.

Proposition 6.5. *Let U and V be distinct summands in a decomposition of $H^{(2^m\,;\,k)}$, defined over a field F, into direct summands. Then $\mathrm{End}_{FS_{2m+k}}(U)$ is commutative and $\mathrm{Hom}_{FS_{2m+k}}(U,V)=0$.*

Proof. Let π_U be the projection map from $H^{(2^m\,;\,k)}$ onto U and let ι_U and ι_V be the inclusion maps of U and V respectively into $H^{(2^m\,;\,k)}$. Suppose that $\phi \in \mathrm{Hom}_{FS_{2n}}(U,V)$ is a non-zero homomorphism. Then $\pi_U\phi\iota_V$ does not commute with $\pi_U\iota_U$. (We compose homomorphisms from left to right.) Moreover sending $\theta \in \mathrm{End}_{FS_{2m+k}}(U)$ to $\pi_U\theta\iota_U$ defines an injective map from $\mathrm{End}_{FS_{2m+k}}(U)$ into the commutative algebra $\mathrm{End}_{FS_{2m+k}}H^{(2^m\,;\,k)}$. □

Proposition 6.5 implies that if λ is a p-regular partition and P^λ is a direct summand of $H^{(2^m\,;\,k)}$, defined over a field of characteristic p, then there are no non-zero homomorphisms from P^λ to any other summand of $H^{(2^m\,;\,k)}$. Thus every composition factor of $H^{(2^m\,;\,k)}$ isomorphic to D^λ must come from P^λ. We also obtain the following corollary.

Corollary 6.6. *Let F be a field of odd characteristic. Given any $w \in \mathbf{N}$ there exists $n \in \mathbf{N}$ and an indecomposable projective module P^λ for FS_n lying in a block of weight w such that $\mathrm{End}_{FS_n}(P^\lambda)$ is commutative.*

Proof. Let γ be the p-core in Lemma 6.3 when $e+1=w$. Taking $k=0$ we see that $w_0(\gamma)=w$. If λ is a maximal element of $\mathcal{E}_0(\gamma)$ then, by Proposition 1.3, P^λ is a direct summand of $H^{(2^m)}$, where $2m=|\lambda|$ and both modules are defined over the field F. The result now follows from Proposition 6.5. □

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Department of Mathematics, Royal Holloway, University of London, United Kingdom

E-mail address: `Eugenio.Giannelli.2011@live.rhul.ac.uk`

E-mail address: `mark.wildon@rhul.ac.uk`