PRIFYSGOL CYMRU ABERTAWE
UNIVERSITY OF WALES SWANSEA

DEGREE EXAMINATIONS 2007

MODULE MAP363

Combinatorics — paper to be sat on 29th May 2007

Time Allowed — 2 hours

There are SIX questions on the paper.
A candidate's best THREE questions will be used for assessment.

No calculators are permitted.

Each question has equal weight. The maximum possible mark is 75/75.

turn over

1. Let G be a connected graph with vertex set V and edge list E.

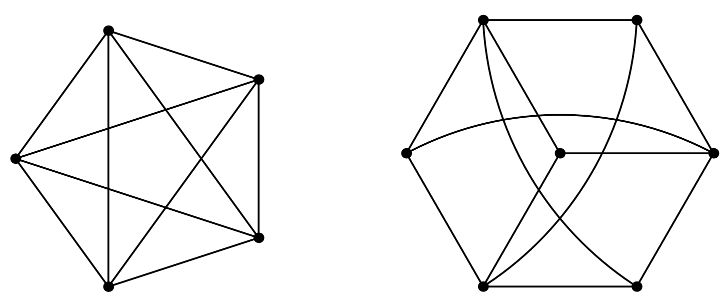
Define the *degree* of a vertex $x \in V$. What does it mean to say that $(x_0, x_1, \ldots, x_m)$ is a *trail* in G? What is the *length* of this trail? When is this trail *closed*? What does it mean to say that G has an *Eulerian trail*? [6 Marks]

State *without proof* a necessary and sufficient condition in terms of vertex degrees for G to have a closed Eulerian trail. [3 Marks]

Using this result, prove that if G has exactly two vertices of odd degree, say x and y, then G has an Eulerian trail. Where must this trail start and finish? [6 Marks]

Show that G has an even number of vertices of odd degree. [*The Handshaking Theorem may be assumed provided it is clearly stated.*] [5 Marks]

For each of the graphs below find with proof the minimum number of continuous pen-strokes required to draw it.



[5 Marks]

2. (a) What is meant by a *closed path* in a graph? What does it mean to say that a graph is *connected*? What does it mean to say that a graph is *simple*? What does it mean to say that a simple graph is a *tree*? [5 Marks]

(b) Prove that a tree on n vertices has exactly $n-1$ edges. [9 Marks]

(c) Show that, up to isomorphism, there are just 2 different trees with 4 vertices. [4 Marks]

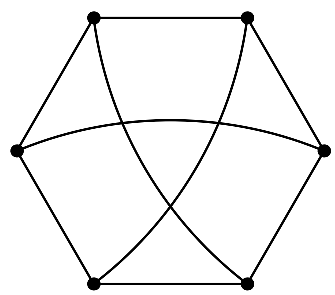
(d) Draw all connected simple graphs on 4 vertices with exactly 4 edges. How many such graphs are there (up to isomorphism)? [7 Marks]

3. What does it mean to say that a simple graph is *planar*? What is meant by a *face* of a planar graph? [4 Marks]

State and prove Euler's formula relating the number of vertices, edges and faces of a simple planar graph. [10 Marks]

[*You may assume the result of* Q2(b), *that a tree on* n *vertices has exactly* $n-1$ *edges.*]

Prove that the graph below is not planar.



[7 Marks]

Now let $K_{m,n}$ be the complete bipartite graph on two sets of sizes m and n. Explain why the last result shows that if $m \geq n \geq 3$ then G is not planar. [4 Marks]

 Turn over.

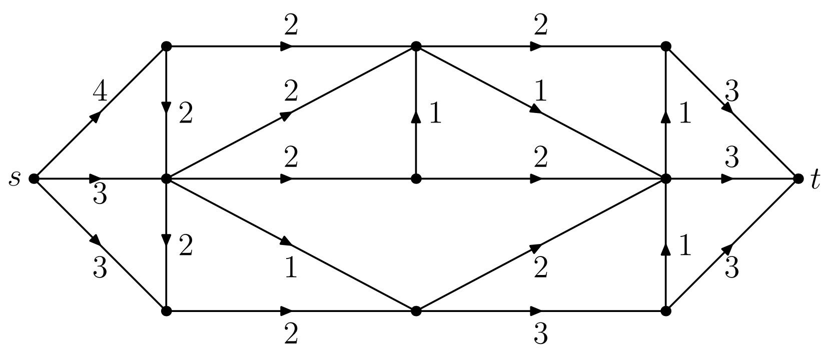
4. Let N be a network with vertex set V and edge set E. Write $c(x,y)$ for the capacity of the edge $(x,y) \in E$. Let $s \in V$ be the source vertex and let $t \in V$ be the target vertex

(a) What is meant by a cut (S,T) of N? Define the *capacity* of the cut (S,T). [3 Marks]

(b) What does it mean to say that f is a flow in N? Define the value of f. [4 Marks]

(c) Prove the the value of any flow from s to t is less than or equal to the capacity of any cut of N. [8 Marks]

(d) Find with proof the maximal flow value in the network below. [10 Marks]



(The numbers show the capacities of the edges.)

5. (a) Let X be a set and let $G \leq \mathrm{Sym}(X)$ be a permutation group acting on X. For $x \in X$ define the *orbit of* x, Orb x. For $g \in G$ define the *fixed point set of* g, Fix g. Prove that

$$\frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix} g|$$

is the number of orbits of G on X. [12 Marks]

[*You may assume the orbit-stabiliser theorem, provided that it is clearly stated.*]

(b) A 6 bead necklace is made using c different colours of beads. Two necklaces should be regarded as the same if one is a rotation of the other. (Reflections should *not* be considered.) Find the number of different necklaces as a polynomial in c. [9 Marks]

If there are three colours, white, grey and black, find the number of different necklaces which have 2 beads of each colour.

[4 Marks]

6. Let $n \in \mathbb{N}_0$. What does it mean to say that $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a *partition* of n? [3 Marks]

Let $f(n)$ be the number of partitions of n into parts of size 5 and 7. Prove that

$$\sum_{n=0}^{\infty} f(n)x^n = \frac{1}{(1-x^5)(1-x^7)}.$$

[10 Marks]

What is the smallest number n for which $f(n) \geq 2$?

[5 Marks]

For $n \in \mathbb{N}_0$, let $g(n)$ be the number of partitions with parts of sizes 5 and 7 whose sum of parts is *at most* n. Find the generating function for g. [7 Marks]