EXPANDED VERSION OF §5

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The argument at the start of the second paragraph of §5 of my paper [2] leaves too much to the reader. Here is a more careful version.

Reminder of setting. The permutation group G acts regularly on the set $\{0,1,\ldots,d-1\}$ and has $\langle g\rangle \cong C_d$ as a regular cyclic subgroup. From §3 we have the corresponding permutation module $M = \langle v_0, v_1, \ldots, v_{d-1}\rangle$, where v_j affords the 1-dimensional representation of $\langle g\rangle$ on which g acts by ζ^j, where ζ is a primitive d-th root of unity. Let ϑ be the complex character of $\langle g\rangle$ defined by $\vartheta(g) = \zeta$. We have seen that

$$M = \langle v_0\rangle \oplus V_1 \oplus \cdots \oplus V_t$$

where each V_k has a basis $\{j : v_j \in B_k\}$ for disjoint $B_k \subseteq \{1,\ldots,d-1\}$. By definition π_k is the character of G afforded by the $\mathbf{C}G$-module V_k. Thus $\pi_k\!\downarrow_{\langle g\rangle} = \sum_{j\in V_k} \vartheta^j$.

Subalgebra. A self-contained proof that the span of the π_k is a subalgebra of the character ring is outlined in my MathOverflow question and my (later) answer, based on [1]: `https://mathoverflow.net/q/319547/7709`.

Details of argument: this may replace the first two paragraph of §5. Since $(a+b)^p \equiv a^p + b^p \bmod p$ for $a, b \in \mathbf{Z}$, we have

$$(\pi_k\!\downarrow_{\langle g\rangle})^p = \Big(\sum_{j\in V_k} \vartheta^j\Big)^p = \sum_{j\in V_k} \vartheta^{jp} + p\phi \tag{1}$$

where ϕ is a character of $\langle g\rangle$. (We do not claim that ϕ is the restriction of a character of G.) Since the linear span of the π_k is a subalgebra of the character ring, we may also write

$$\pi_k^p = a1_G + \sum_\ell (a_\ell + pb_\ell)\pi_\ell$$

for some coefficients $a_\ell \in \{0,1,\ldots,p-1\}$ and $b_\ell \in \mathbf{N}_0$ and $a \in \mathbf{N}_0$. (In the published paper there is a typo at this point: $a1_H$ should be $a1_G$.) Restricting each side to $\langle g\rangle$ we obtain

$$\big(\pi_k\big\downarrow_{\langle g\rangle}\big)^p = a1_{\langle g\rangle} + \sum_\ell (a_\ell + pb_\ell) \sum_{j\in V_k} \vartheta^j. \tag{2}$$

Fix $s \in \{0,1,\ldots,d-1\}$ such that p does not divide s. Let π_ℓ be the unique character in the list $\pi_1,\ldots,\pi_t$ that contains ϑ^s. Since the coefficient of ϑ^s

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in (1) is divisible by p, we see that $a_\ell = 0$. We may therefore write (2) in a better way as

$$(\pi_k\!\downarrow_{\langle g\rangle})^p = a1_{\langle g\rangle} + \sum_{\ell\in L}(a_\ell + pb_\ell)\pi_\ell\!\downarrow_{\langle g\rangle} + p\sum_{\ell\notin L} b_\ell\pi_\ell\!\downarrow_{\langle g\rangle} \tag{3}$$

where L is the set of indices ℓ such that all ϑ^m appearing in π_ℓ have p dividing m. Since the π_ℓ have disjoint support it follows that (3) holds without restriction:

$$\pi_k^p = a1_{\langle g\rangle} + \sum_{\ell\in L}(a_\ell + pb_\ell)\pi_\ell + p\sum_{\ell\notin L} b_\ell\pi_\ell.$$

We may therefore set $\pi = \sum_\ell b_\ell\pi_\ell$ and obtain

$$\pi_k^p - p\pi = a1_G + \sum_{\ell\in L} a_\ell\pi_\ell = a1_G + \sum_{\ell\in L} a_\ell \sum_{j\in B_\ell}\vartheta^j.$$

By definition of the set L, if $a_\ell \neq 0$ then B_ℓ contains only those j with j divisible by p. Hence if $a_\ell \neq 0$ for some ℓ then, by Proposition 3.3, G is imprimitive. We may therefore assume that $a_\ell = 0$ for all ℓ and so

$$\pi_k^p = a1_G + p\pi \tag{4}$$

for some character π of G not containing the trivial character. Comparing (1) and (4) we see that $\sum_{j\in V_k}\vartheta^{jp}$ is equal to some multiple of the trivial character of $\langle g\rangle$, plus p times a character of $\langle g\rangle$. Now take the coefficient of rp for each r with $1 \le r < d/p$ to get that

$$\bigl|\bigl\{j \in B_k : jp \equiv rp \bmod d\bigr\}\bigr|$$

Rest as paper

is a multiple of p for each such r. Identifying $\{0, 1, \ldots, d-1\}$ with $\mathbf{Z}/d\mathbf{Z}$, note that $jp \equiv rp \bmod d$ if and only if $j \in r + \langle d/p\rangle$. Therefore *for each prime p dividing d, each B_k is the union of a subset of $\langle d/p\rangle$ and some proper cosets $r + \langle d/p\rangle$.*

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I thank an anonymous reader (who disclaimed public acknowledgement) for pointing out this gap in the argument. Of course I have full responsibilities for any remaining errors.

References

[1] Wolfgang Knapp, *On Burnside's method*, J. Algebra **175** (1995), no. 2, 644–660.

[2] Mark Wildon, *Permutation groups containing a regular abelian subgroup: the tangled history of two mistakes of Burnside*, Math. Proc. Cambridge Philos. Soc. **168** (2020), no. 3, 613–633.