Lie Algebras (Paper C2.1b)
Hilary Term 2006: Sheet 1

1. Let V be a vector space. Let $\mathsf{gl}(V)$ be the vector space of all linear maps from V to itself with Lie bracket defined by $[x, y] = xy - yx$ for $x, y \in \mathsf{gl}(V)$. Show that $\mathsf{gl}(V)$ is a Lie algebra.

2. (a) If L is a Lie algebra, we define L' to be the linear span of all Lie brackets $[x, y]$ for $x, y \in L$. Show that L' is an ideal of L. (L' is known as the *derived algebra* of L.)

(b) Find the derived algebra of $\mathbb{R}^3_\wedge$ (this was defined in Example 1.5 from lectures). Find also the derived algebra of $\mathsf{b}_2(\mathbb{R})$, the Lie algebra of 2×2 upper-triangular real matrices. Are these Lie algebras isomorphic?

3. Recall that if S is an $n \times n$ matrix with entries in a field F we defined

$$\mathsf{gl}_S(F) = \{x \in \mathsf{gl}_n(F) : x^t S = -Sx\}.$$

(a) Show that $\mathsf{gl}_S(F)$ is a Lie subalgebra of $\mathsf{gl}_n(F)$.

(b) Find a vector space basis for the image of $\mathrm{ad} : \mathbb{R}^3_\wedge \to \mathsf{gl}(\mathbb{R}^3)$. Hence find a matrix T such that $\mathbb{R}^3_\wedge \cong \mathsf{gl}_T(\mathbb{R})$.

(c) Let S be the $2m \times 2m$ matrix

$$\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

Find conditions for a matrix to lie in $\mathsf{gl}_S(\mathbb{C})$ and hence determine the dimension of $\mathsf{gl}_S(\mathbb{C})$.

4. Let F be a field and let $L = \mathsf{gl}_n(F)$. Let $x \in \mathsf{gl}_n(F)$ be a diagonal matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. By describing a basis of eigenvectors for $\mathrm{ad}\, x : L \to L$ show that $\mathrm{ad}\, x$ is diagonalisable, with eigenvalues $\lambda_i - \lambda_j$ for $1 \leq i, j \leq n$.

5. (a) Suppose that L is a 3-dimensional complex Lie algebra with L' of dimension 1. Suppose also that $L' \subseteq Z(L)$. Determine the structure constants of L with respect to a suitable basis and show that up to isomorphism there is a unique such algebra. (This Lie algebra is known as the *Heisenberg algebra*.)

(b) ($\star$) (Optional harder question, needs some bilinear algebra.) Classify up to isomorphism all Lie algebras L such that $\dim L' = 1$ and $L' = Z(L)$.

6. Let L and M be Lie algebras and $\varphi : L \to M$ a surjective Lie homomorphism. Give proofs or counterexamples as appropriate to the following statements:

(i) $\varphi(L') = M'$;

(ii) $\varphi(Z(L)) = Z(M)$;

(iii) if $h \in L$ and $\mathrm{ad}\, h : L \to L$ is diagonalizable then $\mathrm{ad}\, \varphi(h) : M \to M$ is diagonalizable.

What changes if φ is an isomorphism?

Lie Algebras (Paper C2.1b)
Hilary Term 2006: Sheet 2

7. Let $n \geq 2$. Show that the trace map, $\mathrm{tr} : \mathsf{gl}_n(\mathbb{C}) \to \mathbb{C}$ is a Lie algebra homomorphism. (Here $\mathbb{C}$ should be regarded as the 1-dimensional abelian Lie algebra.) Describe explicitly the kernel of tr and the elements of the quotient space $\mathsf{gl}_n(\mathbb{C})/\ker \mathrm{tr}$.

8. Find the structure constants of $\mathsf{sl}_2(\mathbb{C})$ with respect to the basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Show that the only ideals of $\mathsf{sl}_2(\mathbb{C})$ are 0 and itself.

9. If a Lie algebra L is a vector space direct sum of two Lie subalgebras L_1 and L_2 such that $[L_1, L_2] = 0$, then we say that L is the *direct sum* of L_1 and L_2 and write $L = L_1 \oplus L_2$.

(a) Show that $\mathsf{gl}_2(\mathbb{C})$ is the direct sum of $\mathsf{sl}_2(\mathbb{C})$ with the subalgebra of scalar multiples of the 2×2 identity matrix.

(b) Show that if L is the direct sum of Lie subalgebras L_1 and L_2 then L_1 and L_2 are in fact ideals of L. Show also that $Z(L) = Z(L_1) \oplus Z(L_2)$ and $L' = L_1' \oplus L_2'$.

(c) Which of the 3-dimensional complex Lie algebras L with $\dim L' \leq 1$ admit a non-trivial direct sum decomposition?

(d) Are the summands in the direct sum decomposition of a Lie algebra uniquely determined? That is, if $L = L_1 \oplus L_2$ and $L = M_1 \oplus M_2$, must $\{L_1, L_2\} = \{M_1, M_2\}$?

10. Let $L = \langle t \rangle \oplus V$ where V is a 2-dimensional complex vector space. Let $T : V \to V$ be an invertible linear transformation. Define a Lie bracket on L by

$$[v, w] = 0, \ [t, v] = T(v) \quad \text{for all } v, w \in V$$

and extending linearly. Check that this defines a Lie algebra and that $L' = V$. For non-zero $\lambda \in \mathbb{C}$ let L_λ be the Lie algebra obtained when

$$T = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}.$$

Show that $L_\lambda \cong L_\mu$ if and only if $\lambda = \mu$ or $\lambda = \mu^{-1}$.

11. Let $L = \mathsf{b}_n(F)$, the Lie algebra of upper-triangular $n \times n$ matrices over a field F. Find the derived series of L, verifying your answer for the case $n = 4$. Deduce that L is solvable and determine the least m for which $L^{(m)} = 0$.

12. ($\star$) (Optional question.) Let S and T be matrices with entries in a field F. Suppose that S and T are congruent; that is, $P^t S P = T$ for some invertible matrix P. Prove that $\mathsf{gl}_S(F)$ is isomorphic to $\mathsf{gl}_T(F)$. (See Question 3 on sheet 1 for the definition of $\mathsf{gl}_S(F)$.)

Lie Algebras (Paper C2.1b)
Hilary Term 2006: Sheet 3

13. Let V be a vector space and let $L = \mathsf{gl}(V)$.

(a) Show that

$$(\operatorname{ad} x)^m y = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} x^k y x^{m-k} \quad \text{for all } x, y \in L.$$

Deduce that if $x \in L$ is nilpotent then $\operatorname{ad} x : L \to L$ is also nilpotent. Does the converse hold?

(b) Any $x \in L$ may be written in the form $x = d + n$ where $d \in L$ is diagonalisable, $n \in L$ is nilpotent and d and n commute; this is known as the *Jordan decomposition* of x. Show that $\operatorname{ad} d : L \to L$ is diagonalisable and $\operatorname{ad} n : L \to L$ is nilpotent. Deduce that $\operatorname{ad} x$ has Jordan decomposition $\operatorname{ad} x = \operatorname{ad} d + \operatorname{ad} n$.

14. (a) Show that if a Lie algebra L has an ideal I such that both I and L/I are solvable, then L is solvable.

(b) Let I and J be ideals of a Lie algebra L. Let $[I, J]$ be the span of all Lie brackets $[x, y]$ with $x \in I$ and $y \in J$. Show that $[I, J]$ is an ideal of L.

(c) Use part (b) to show that if L is a Lie algebra with a non-zero radical then L has a non-zero abelian ideal. Deduce that a Lie algebra is semisimple if and only if it has no abelian ideals.

15. Let L be the set of complex matrices of the form $\begin{pmatrix} \alpha & \beta & \lambda \\ \gamma & \delta & \mu \\ 0 & 0 & 0 \end{pmatrix}$ where $\alpha + \delta = 0$.

Show that L is a Lie subalgebra of $\mathsf{gl}_3(\mathbb{C})$. Find the radical of L and show that L contains a subalgebra isomorphic to $L/\operatorname{rad} L$. Prove that the only ideal of L strictly contained in $\operatorname{rad} L$ is $\{0\}$.

16. Let F be a field of characteristic not 3. Let L be a Lie algebra over F such that $(\operatorname{ad} t)^2 = 0$ for all $t \in L$. By expanding $[x + y, [x + y, z]]$ show that

$$[y, [x, z]] = -[x, [y, z]] \quad \text{for all } x, y, z \in L.$$

Hence use the Jacobi identity to show that $L^2 = [L, [L, L]] = \{0\}$. ($\star$) What can be said if F has characteristic 3?

17. Let V be a complex vector space. Suppose that $L \subseteq \mathsf{gl}(V)$ is an abelian Lie algebra. Show that there is a basis of V in which all the elements of L are represented by upper-triangular matrices.

Lie Algebras (Paper C2.1b)
Hilary Term 2006: Sheet 4

18. Let L be a Lie algebra. Show that the following conditions are equivalent

(i) $L^m = 0$;

(ii) there is a chain of ideals of L,

$$L = I_0 \supseteq I_1 \supseteq \ldots \supseteq I_m = \{0\}$$

such that $I_{k-1}/I_k \leq Z(L/I_k)$ for $1 \leq k \leq m$;

(iii) $\operatorname{ad} x_1 \circ \operatorname{ad} x_2 \circ \ldots \circ \operatorname{ad} x_m = 0$ for all $x_1 \ldots, x_m \in L$.

19. Give an example of a Lie algebra L and an ideal I of L such that I and L/I are nilpotent but L is not. ($\star$) Show that the sum of two nilpotent ideals of a Lie algebra is nilpotent; note that the method used in the solvable case cannot be applied here.

20. Show that a complex Lie algebra is nilpotent if and only if all its 2-dimensional Lie subalgebras are abelian. *Hint*: Use the second version of Engel's Theorem.

21. Let L be a complex Lie algebra. Use Lie's Theorem to prove that L is solvable if and only if L' is nilpotent.

22. Let V be a complex vector space and let $L \subseteq \mathsf{gl}(V)$ be isomorphic to the 2-dimensional non-abelian Lie algebra. Prove (without using Lie's Theorem) that V contains a common eigenvector for the elements of L. *Hint*: use the result from lectures that if x and y are linear maps on V such that $[x, y]$ commutes with x then $xy - yx$ is nilpotent.

23. Let p be prime and let F be a field of characteristic p. Let $L = \langle x, y \rangle$ with Lie bracket defined by $[x, y] = x$ be a w2-dimensional non-abelian Lie algebra over F. Let V be a p-dimensional vector space over F. Show that the map $\varphi : L \to \mathsf{gl}(V)$ defined by

$$\varphi(x) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad \varphi(y) = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & p-2 & 0 \\ 0 & 0 & \ldots & 0 & p-1 \end{pmatrix}$$

is a faithful representation of L. Show that $\varphi(x)$ and $\varphi(y)$ have no common eigenvector, and deduce that Lie's Theorem may fail in prime characteristic. ($\star$) Show that in fact V is an irreducible representation of L; that is, V has no non-trivial proper subspace invariant under $\varphi(L)$.

Lie Algebras (Paper C2.1b)
Hilary Term 2006: Sheet 5

24. (a) Let L be a Lie algebra and let $\varphi : L \to \mathsf{gl}(V)$ be a representation of L. Given $w \in V$, define the subrepresentation *generated by* w to be the subspace W of V spanned by all elements of the form

$$\varphi(x_1)\dots\varphi(x_m)(w)$$

where $x_1 \dots, x_m \in L$ and $m \geq 0$. Show that W is the smallest subrepresentation of V which contains w. Prove that V is irreducible if and only if V is equal to the subrepresentation generated by any of its non-zero elements.

(b) Show that the adjoint representation of a non-zero Lie algebra L is irreducible if and only if L has no non-trivial proper ideals.

(c) Prove that the natural representation of $\mathsf{sl}_n(\mathbb{C})$ is irreducible for $n \geq 1$.

25. Let L be a complex Lie algebra and let V be an L-module. Show that if $z \in Z(L)$ then the map $\theta_z : V \to V$ defined by $\theta_z(v) = z \cdot v$ is an L-module homomorphism. Hence show that if L has a faithful irreducible representation then $\dim Z(L) \leq 1$.

26. Let F be a field and let $L = \mathsf{b}_n(F)$. Let $V = F^n$ be the natural L-module.

(a) Let $e_1, \dots, e_n$ be the standard basis of F^n. For $1 \leq r \leq n$, let $W_r = \mathrm{Span}\{e_1, \dots, e_r\}$. Prove that W_r is a submodule of V.

(b) Show that every non-zero submodule of V is equal to one of the W_r. Deduce that if $n \geq 2$ then V cannot be written as a direct sum of irreducible L-modules.

27. (a) Find an explicit isomorphism between V_2 and the adjoint representation of $\mathsf{sl}_2(\mathbb{C})$.

(b) Let L be the Lie algebra of complex matrices of the form $\begin{pmatrix} \alpha & \beta & \lambda \\ \gamma & \delta & \mu \\ 0 & 0 & 0 \end{pmatrix}$, and let $M \subseteq L$ be the Lie subalgebra of matrices where $\lambda = \mu = 0$ and $\alpha + \delta = 0$. Show that L may be regarded as a representation of M via the adjoint action and that L decomposes as a direct sum of irreducible representations of M.

28. Let $L = \langle x, y, z \rangle_{\mathbb{C}}$ be the complex Heisenberg algebra with its usual generators, so $[x, y] = z$ and z is central. Suppose that $\varphi : L \to \mathsf{gl}(V)$ is a 2-dimensional representation of L. Use Lie's Theorem to prove that $\varphi(z) = 0$ and deduce that φ is not faithful. Show that L does have a faithful 3-dimensional representation.

29. Let L be a Lie algebra. A *derivation* of L is a linear map $D : L \to L$ such that $D([x, y]) = [D(x), y] + [x, D(y)]$ for all $x, y \in L$.

(a) Show that if $x \in L$ then $\operatorname{ad} x : L \to L$ is a derivation of L.

(b) Show that if D and E are derivations of a Lie algebra L then $DE - ED$ is also a derivation. Hence show that if $\operatorname{Der} L$ is the set of derivations of L then $\operatorname{Der} L$ is a Lie subalgebra of $\mathsf{gl}(L)$ containing $\operatorname{ad} L$ as an ideal. Need DE be a derivation?

Lie Algebras (Paper C2.1b)
Hilary Term 2006: Sheet 6

30. Let U be a finite-dimensional module for $\mathsf{sl}_2(\mathbb{C})$. Suppose that $U = S_1 \oplus \ldots \oplus S_k$ where each S_i is a simple $\mathsf{sl}_2(\mathbb{C})$-submodule of U. Show that the number k of simple summands is equal to $\dim U_0 + \dim U_1$, where for $\lambda \in \mathbb{C}$,

$$U_\lambda = \{v \in U : h \cdot v = \lambda v\}.$$

31. Let $L = \mathsf{sl}_3(\mathbb{C})$. Let $M \cong \mathsf{sl}_2(\mathbb{C})$ be the subalgebra of L defined by

$$M = \left\{ \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & 0 \end{pmatrix} : \alpha + \delta = 0 \right\}.$$

Let $h = e_{11} - e_{22} \in M$. Find a basis for L consisting of eigenvectors for h. Consider L as an M-module via the adjoint map. Express L as a direct sum of irreducible M-modules.

32. (a) Let V be a vector space with basis $v_1, \ldots, v_n$. Suppose that $(-,-)$ is a symmetric bilinear form on V, with matrix A in the basis $v_1, \ldots, v_n$, so $A_{ij} = (v_i, v_j)$. Show that $V^\perp = 0$ if and only the matrix A is non-singular.

(b) Show that if L is a nilpotent Lie algebra then the Killing form on L is identically zero.

(c) ($\star$) Suppose that L is a 3-dimensional complex solvable Lie algebra that is not nilpotent. Using the classification given in lectures find the possible Killing forms for L. Deduce that the converse to (b) does not hold.

(d) Compute the Killing form of $\mathsf{sl}_2(\mathbb{C})$ and check that it is non-degenerate. Is the Killing form of $\mathsf{gl}_2(\mathbb{C})$ non-degenerate?

33. Let L be a semisimple Lie algebra. Suppose that $L = L_1 \oplus \ldots \oplus L_r$ where the L_r are ideals which are simple. Show that $Z(L) = 0$ and $L' = L$. Now suppose that J is a simple ideal of L. By considering $[L, J]$ show that $J = L_i$ for some i.

34. (a) Let $L = \mathsf{so}_3(\mathbb{C})$. Find the root space decomposition of L with respect to its 1-dimensional subalgebra of diagonal matrices. Hence show that $\mathsf{so}_3(\mathbb{C}) \cong \mathsf{sl}_2(\mathbb{C})$.

(b) Let $L = \mathsf{sl}_n(\mathbb{C})$ where $n \geq 2$. Let H be the subalgebra of diagonal matrices in L and let $\lambda_i \in H^\star$ be the map sending the diagonal matrix d to its entry in position i. Show that the root space decomposition of L with respect to H is

$$L = H \oplus \bigoplus_{i \neq j} \langle e_{ij} \rangle$$

and find the corresponding roots in $H^\star$ in terms of the λ_i. Hence show that L is a simple Lie algebra and that H is a Cartan subalgebra of L.

Hint: if I is a non-zero ideal of L then, as H acts diagonalisably on I, I must contain a common eigenvector for the elements of H, so either $I \subseteq H$ or $e_{ij} \in I$ for some $i \neq j$.

35. ($\star$) Recall that if L is a Lie algebra then a linear map $D : L \to L$ is a *derivation* if $D([x, y]) = [D(x), y] + [x, D(y)]$ for all $x, y \in L$.

(a) Show that if I is an ideal of a Lie algebra L and $x \in L$ then $\operatorname{ad} x : I \to I$ is a derivation of I.

(b) Find an example of a Lie algebra L with an ideal I and a derivation $D : I \to I$ that is not of the form $\operatorname{ad} x$ for any $x \in I$.

(c) Let L be a semisimple Lie algebra and let I be a non-zero ideal of L of the smallest possible dimension. Show that I is a simple Lie algebra. Let B be the *centraliser* of I in L; that is,

$$B = \{x \in L : [x, a] = 0 \text{ for all } a \in I\}.$$

Show that B is an ideal of L and that $L = I \oplus B$. Hence show that a semisimple Lie algebra is a direct sum of simple Lie algebras.

In part (c) you may assume that an ideal of a semisimple algebra is semisimple, and that if L is a semisimple Lie algebra and $D : L \to L$ is a derivation then $D = \operatorname{ad} x$ for some $x \in L$.

Lie Algebras (Paper C2.1b)
Hilary Term 2006: Sheet 7

36. Suppose that L is a complex semisimple Lie algebra with Cartan subalgebra H. Let Φ be the set of roots of L with respect to H. Use results from lectures to prove that

$$\dim L = \dim H + |\Phi|$$

and that $|\Phi|$ is even. Hence show that there are no complex semisimple Lie algebras of dimensions 4, 5, or 7. Give examples of complex semisimple Lie algebras of dimensions 6 and 8.

37. Say that a Lie subalgebra H of a complex Lie algebra L is *toral* if all its elements are semisimple. Show that a toral subalgebra is automatically abelian. *Hint*: show that if $x \in H \backslash Z(H)$ then $\operatorname{ad} x$ has a eigenvector $y \in H$ with a non-zero eigenvalue. Now look at $\operatorname{ad} y$ to get a contradiction.

Deduce that a maximal toral subalgebra of L is a Cartan subalgebra of L.

38. Let L be a complex semisimple Lie algebra with Cartan subalgebra H and let Φ be the set of roots of L with respect to H.

(a) Show that if $\beta(h) = 0$ for all $\beta \in \Phi$ then $h \in Z(L)$. Deduce that $h = 0$.

(b) In the main step in the proof of Theorem 8.8, we showed that if $x \in L_\alpha$ and $y \in L_{-\alpha}$, and $h = [x, y] \neq 0$, then $\alpha(h) \neq 0$. Here is an alternative proof of this using root string modules. Suppose that $\alpha(h) = 0$. Let $\beta \in \Phi$ be any root. Let

$$U = \bigoplus_c L_{\beta+c\alpha}$$

be the α-root string module through β, By considering the trace of h on U, show that $\beta(h) = 0$ and hence obtain a contradiction.

39. Recall that $\mathsf{sp}_4(\mathbb{C}) = \mathsf{gl}_S(\mathbb{C})$ where $S = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$. (See question 3 on sheet 1 for the definition of $\mathsf{gl}_S(\mathbb{C})$.)

(a) Determine the root space decomposition of $\mathsf{sp}_4(\mathbb{C})$ with respect to its Cartan subalgebra H of diagonal matrices.

(b) Let Φ be the set of roots of $\mathsf{sp}_4(\mathbb{C})$ with respect to H. Show that Φ contains roots α, β such that $(\beta, \beta)/(\alpha, \alpha) = 2$, $(\beta, \alpha) < 0$ and $2(\beta, \alpha)/(\alpha, \alpha) \times 2(\alpha, \beta)/(\beta, \beta) = 2$.

(c) Draw a diagram showing the elements of Φ in the real subspace of $H^\star$ spanned by the roots. What are the lengths of the root strings?

40. Let L be a complex semisimple Lie algebra with Cartan subalgebra H and roots Φ. Suppose that Φ may be decomposed in a non-trivial way as $\Phi = \Phi_1 \cup \Phi_2$ where $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1, \beta \in \Phi_2$. Show that L is a direct sum of two smaller dimensional Lie algebras. Does the converse hold?

41. ($\star$) Let $L = \mathsf{sl}_n(\mathbb{C})$ and let H be the Cartan subalgebra of L consisting of all diagonal matrices. Show that there is an element $h \in H$ such that $C_L(h) = H$. Show moreover that h may be chosen so that the eigenspaces of $\operatorname{ad} h$ are exactly the root spaces of H.

MJW, October 17, 2006