# Prerequisites from Linear Algebra

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It is asssumed that the reader is familiar with the linear and bilinear algebra from the second year core algebra course. The main 'extra' that is needed is the idea of a quotient vector space; this will be familiar to those who have done B2a algebra, but maybe not to others, so I summarise it below. It will also be useful to know the statement of Jordan normal form. We really only need one result from bilinear algebra, with which I think everyone will be familiar, but it's repeated below just to make sure.

Highly recommended for alternating or additional reading is Halmos' book, *Finite-Dimensional Vector Spaces*.

## 1 Quotient Spaces and isomorphism theorems

Suppose that $W$ is a subspace of the vector space $V$. A *coset of* $W$ is a set of the form

$$v + W = \{v + w : w \in W\}.$$

It is important to realise that unless $W = 0$, each coset will have many different labels; in fact, $v + W = v' + W$ if and only if $v - v' \in W$.

The *quotient space* $V/W$ is the set of all cosets of $W$. This becomes a vector space, with zero element $0 + W = W$, if addition is defined by

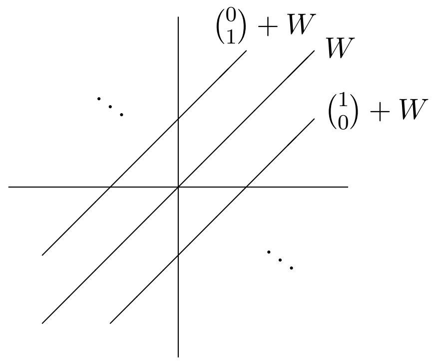
$$(v + W) + (v' + W) = (v + v') + W \quad \text{for } v, v' \in V$$

and scalar multiplication by

$$\lambda(v + W) = \lambda v + W \quad \text{for } v, v' \in V,\ \lambda \in F.$$

One must check that these operations are *well-defined*; that is, they do not depend on the choice of labelling elements. Suppose for instance that $v + W = v' + W$. Then, since $v - v' \in W$, we have $\lambda v - \lambda v' \in W$ for any scalar $\lambda$, so $\lambda v + W = \lambda v' + W$.

The following diagram shows the elements of $\mathbb{R}^2/W$, where $W$ is the subspace of $\mathbb{R}^2$ spanned by $\binom{1}{1}$.




The cosets $\mathbb{R}^2/W$ are all the translations of the line $W$. One can choose a standard set of coset representatives by picking any line through 0 (other than $W$) and looking at its intersection points with the cosets of $W$; this gives a geometric interpretation of the isomorphism $\mathbb{R}^2/W \cong \mathbb{R}$.

It is often useful to consider quotient spaces when attempting a proof by induction on the dimension of a vector space. In this context, it can be useful to know that if $v_1, \ldots, v_k$ are vectors in $V$ such that the cosets $v_1 + W, \ldots, v_k + W$ form a basis for the quotient space $V/W$, then $v_1, \ldots, v_k$, together with any basis for $W$, forms a basis for $V$.

We can now state the isomorphism theorems for vector spaces.

**Theorem 1.1.** (a) *Let $V$ and $W$ be vector spaces and let $x : V \to W$ be a linear map. Then* $\ker x$ *is a subspace of $V$,* $\operatorname{im} x$ *is a subspace of $W$, and*

$$V/\ker x \cong W.$$

*Now let $U$ and $W$ be subspaces of $V$.* (b) $(U+W)/W \cong U/(U \cap W)$. (c) *The quotient space $W/U$ is a subspace of $V/U$ and* $(V/U)/(W/U) \cong V/W$.

*Proof.* (a) Define a map $\phi : V/\ker x \to \operatorname{im} x$ by

$$\phi(v + \ker x) = x(v).$$

This map is well-defined since if $v + \ker x = v' + \ker x$ then $v - v' \in \ker x$, so $\phi(v + \ker x) = x(v) = x(v') = \phi(v' + \ker x)$. It is routine to check that $\phi$ is linear, injective, and surjective, so it gives the required isomorphism.

For (b) consider the composite of the inclusion map $U \to U + W$ with the quotient map $U + W \to (U + W)/W$. This gives us a linear map $U \to (U + W)/W$. Under this map, $x \in U$ is sent to $0 \in (U + W)/W$ if and only if $x \in W$, so its kernel is $U \cap W$. Now apply part (a). Part (c) can be proved similarly; we leave this to the reader. $\square$

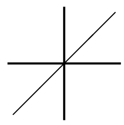
## 2 Interlude: The Diagonal Fallacy

Consider the following (fallacious) argument. Let $V$ be a 2-dimensional vector space, say with basis $v_1$, $v_2$. Let $x : V \to V$ be the linear map whose matrix with respect to this basis is

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

We claim that if $U$ is an $x$-invariant subspace of $V$; that is, $x(U) \subseteq U$, then either $U = 0$, $U = \mathrm{Span}\{v_1\}$, or $U = V$. Clearly each of these subspaces is invariant under $x$, so we only need to prove that there are no others. But since $x(v_2) = v_1$, $\mathrm{Span}\{v_2\}$ is not $x$-invariant. (QED?)

Here we committed the *diagonal fallacy*: We assumed that an arbitrary subspace of $V$ would contain one of our chosen basis vectors. This assumption is very tempting — which perhaps explains why it is so often made[1] — but it is nonetheless totally unjustified. I suspect one reason why people end up committing this error is that they get confused with a *good* way to use linearity, namely the fact that one can save time and space by only defining a linear map on elements of a given basis.



## 3 Jordan Normal Form

Let $V$ be a finite-dimensional complex vector space and let $x : V \to V$ be a linear map. The exercise below outlines a proof that one can always find a basis of $V$ in which $x$ is represented by an upper triangular matrix. For many purposes, this result is sufficient. For example, since the eigenvalues of a matrix in upper triangular form are its diagonal entries, it implies that a nilpotent map may be represented by a strictly upper triangular matrix, and so nilpotent maps have trace 0.

**Exercise 3.1.** *Let $V$ be an $n$-dimensional vector space where $n \geq 1$, and let $x : V \to V$ be a linear map.*

(i) *Show that $x$ has an eigenvector, $v$ say.*

(ii) *Let $U = \mathrm{Span}\{v\}$. Show that $x$ induces a linear transformation $\bar{x} : V/U \to V/U$. By induction, we know that there exists a basis $\{v_1 + U \dots v_{n-1} + U\}$ of $V/U$ in which $\bar{x}$ has a upper triangular matrix. Prove that $\{v, v_1, \dots, v_{n-1}\}$ is a basis of $V$ and that the matrix of $x$ in this basis is upper triangular.*

[1] The author's personal record is hearing the diagonal fallacy committed in three tutorials in a row, on two different courses. After reading Kafka's *The Penal Colony* an unpalatable but probably highly successful to the problem occured to him. (See marginal diagram.)

Sometimes, however, one needs the full strength of Jordan normal form. A general matrix in Jordan normal form looks like

$$\begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_r \end{pmatrix},$$

where each $A_i$ is a *Jordan block matrix* $J_t(\lambda)$ for some $t \in \mathbb{N}$ and $\lambda \in \mathbb{C}$:

$$J_t(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix}_{t \times t.}$$

Any linear transformation of a complex vector space can be represented by a matrix in Jordan normal form. One can successfully use Jordan normal form without knowing anything about how to prove this; that said, if you really want a proof you might see `www.maths.ox.ac.uk/~wildon/JNF.pdf`.

# 4 Bilinear Algebra

**Definition 4.1.** *A* bilinear form *on a vector space $V$ is a map*

$$(-,-) : V \times V \to F$$

*such that*

$$\begin{aligned} (\lambda_1 v_1 + \lambda_2 v_2, w) &= \lambda_1 (v_1, w) + \lambda_2 (v_2, w), \\ (v, \mu_1 w_1 + \mu_2 w_2) &= \mu_1 (v, w_1) + \mu_2 (v, w_2), \end{aligned}$$

*for all $v_i, w_i \in V$ and $\lambda_i, \mu_i \in F$.*

For example, if $F = \mathbb{R}$ and $V = \mathbb{R}^n$, then the usual dot product is a bilinear form on $V$.

Given a subset $U$ of a vector space $V$, we set

$$U^\perp = \{v \in V : (u, v) = 0 \text{ for all } u \in U\}.$$

This is always a subspace of $V$. We say that the form $(-,-)$ is *non-degenerate* if $V^\perp = \{0\}$.

There is an important connection between bilinear forms and dual spaces. Let $\varphi : V \to V^\star$ be the linear map defined by $\varphi(v) = (-, v)$. That is, $\varphi(v)$ is the linear map sending $u \in V$ to $(u, v)$. If $(-,-)$ is non-degenerate, then $\ker \varphi = 0$, so by dimension counting, $\varphi$ is an isomorphism from $V$ to $V^\star$. Hence every element of $V^\star$ is of the form $(-, v)$ for a unique $v \in V$; this is a special case of the *Riesz representation theorem.*