MT182 Matrix Algebra: Sheet 1

Attempt at least questions 1 to 5. Please staple your answers together and put your name and student number on the top sheet. Do not return the problem sheet.

The lecturer will be happy to discuss any of the questions in office hours.

Feedback. Please write down something from lectures that (a) you enjoyed or found clear, *and/or* (b) is confusing to you, despite some thought.

To be handed in at the 10am lecture on Friday 22nd January.

1. (a) Start at $(1, 2, 3)$ and apply the displacement vector $\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$. What is the finishing point?

(b) Write down the displacement vector from $(1, 2, 3)$ to $(-3, 4, 0)$.

(c) If you apply the displacement vector in (b) starting at the point $(2, 0, 13)$, what is the finishing point?

(d) With the same displacement vector, which starting point will produce the finishing point $(20, 1, 3)$?

2. (a) Show that the points $A = (1, 7, 0)$, $B = (1, 8, 2)$, $C = (1, 9, 4) \in \mathbb{R}^3$ lie on a straight line. [*Hint:* Consider the displacement vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.]

(b) Find $z \in \mathbb{R}$ such that the point $(1, 0, z)$ is also on this line.

3. Let

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}.$$

(a) Compute $(\mathbf{a} + \mathbf{b})/2$, $(\mathbf{a} + \mathbf{c})/4$, $(\mathbf{b} + \mathbf{c})/6$.

(b) Given x, y, $z \in \mathbb{R}$, find λ, μ, $\nu \in \mathbb{R}$, in terms of x, y, z, such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{c}.$$

4. (a) Let Π be the plane containing the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Find a vector $\mathbf{n} \in \mathbb{R}^3$ and a scalar $\alpha \in \mathbb{R}$ such that

$$\Pi = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} \cdot \mathbf{n} = \alpha\}.$$

(b) Let ℓ be the line through the points $(-2, 0, 1)$ and $(0, 1, 2)$. There is a unique point P contained in both ℓ and Π. Write P as a vector.

(c) Let θ be the angle between ℓ and Π. Find $\cos \theta$, giving your answer in as simple a form as possible. Write θ in degrees to three decimal places.

5. Let

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \ \mathbf{c} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

(a) Show that if $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ can be written as $\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$, then $x + y + z = 0$.

(b) Find a vector $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ that *cannot* be written as a linear combination of $\mathbf{a}, \mathbf{b}, \mathbf{c}$. (Equivalently: find $\mathbf{v} \in \mathbb{R}^3$ such that the equation $\mathbf{v} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$ has no solution with $\lambda, \mu, \nu \in \mathbb{R}$.)

Bonus questions

6. A boat has a top speed of 20km/h (kilometres per hour). A river is 2km wide and the water flows at 10km/h.

(a) If you set off from one bank and sail perpendicular to the current, how far downstream will you land? How long will your crossing take?

(b) Suppose you want to land opposite your starting point. What direction should you sail in? How long will your crossing take?

7. Let ABC be a triangle. The *midpoint* of the edge BC is the point half-way between B and C. The *median* of A the line through A and the midpoint of BC. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the vectors corresponding to the points A, B and C.

(a) Show that the midpoint of BC is $\frac{1}{2}(\mathbf{b} + \mathbf{c})$.

(b) Show that $\mathbf{v}$ is on the median of A if and only if

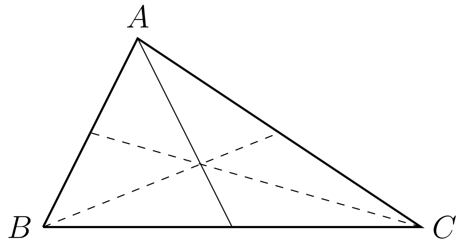
$$\mathbf{v} = \lambda\mathbf{a} + \tfrac{1}{2}(1 - \lambda)(\mathbf{b} + \mathbf{c})$$

for some $\lambda \in \mathbb{R}$. (The median includes points not inside the triangle.)

(c) Find similar conditions for $\mathbf{v}$ to be on the medians of B and C.

(d) Deduce that the medians of A, B and C meet at $\frac{1}{3}(\mathbf{a} + \mathbf{b} + \mathbf{c})$.

(e) What is the physical interpretation of (d)?



MT182 Matrix Algebra: Sheet 2

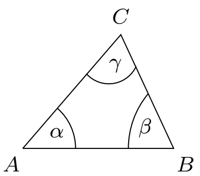
Attempt at least questions 1 to 5. Please staple your answers together and put your name and student number on the top sheet. Do not return the problem sheet.

The lecturer will be happy to discuss any of the questions in office hours.

Feedback. Please write down something from lectures that (a) you enjoyed or found clear, *and/or* (b) is confusing to you, despite some thought.

To be handed in at the 10am lecture on Friday 29th January.

1. Let A, B, $C \in \mathbb{R}^3$ be the points with coordinates $(1, 7, 3)$, $(4, 5, 2)$, $(2, 4, 5)$.



(a) Write down the displacement vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$.

(b) What is $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$? Explain in one equation or one sentence why no calculation is needed to see this.

(c) Find the angle α between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. [*Hint:* $\overrightarrow{AC} = -\overrightarrow{CA}$.]

(d) Find another angle in the triangle ABC. What can you conclude?

(e) How could this be seen in a quicker way? [*Hint:* what are the sides of ABC?]

2. The vectors $\mathbf{a}$ and $\mathbf{b}$ satisfy

$$||\mathbf{a}|| = 13, \ ||\mathbf{b}|| = 19, \ ||\mathbf{a} + \mathbf{b}|| = 24\,.$$

Find $\mathbf{a} \cdot \mathbf{b}$ and $||\mathbf{a} - \mathbf{b}||$. [*Hint:* Consider $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$. Use Lemma 1.17.]

3. (a) Let $\mathbf{a} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$, and $\mathbf{b} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$. Compute $\mathbf{a} \times \mathbf{b}$, and verify that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ by computing appropriate dot products.

(b) Show that $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ for all $\mathbf{u}$, $\mathbf{v} \in \mathbb{R}^3$.

4. (a) Let $A = (1, 2, 3) \in \mathbb{R}^3$. Find the distance of A from the plane

$$\Pi = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{v} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 0\}.$$

Find a point $P \in \Pi$ such that $||\overrightarrow{AP}||$ is minimized. What is this minimum distance?

(b) Find the distance of $A = (1, -1, 0)$ from the plane through $\mathbf{i}$ with normal direction $\mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{k}$. [*Hint:* translate A and the plane by $-\mathbf{i}$.]

Please turn over

5. (a) Find vectors $\mathbf{a}$, $\mathbf{c} \in \mathbb{R}^3$ such that the line through the points $(1, 2, 2)$ and $(0, 2, 1)$ is $\{\mathbf{a} + \lambda\mathbf{c} : \lambda \in \mathbb{R}\}$.

   (b) Similarly, find vectors $\mathbf{a}'$, $\mathbf{c}' \in \mathbb{R}^3$ such that the line through the points $(1, 2, 1)$ and $(0, 4, 1)$ is $\{\mathbf{a}' + \lambda'\mathbf{c}' : \lambda' \in \mathbb{R}\}$.

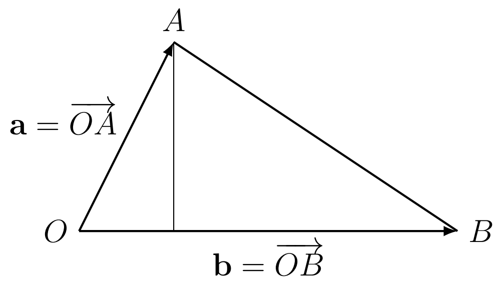
   (c) Find the closest distance between two points P and P' with P on the first line and P' on the second. Do the two lines meet?

Bonus questions

6. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$ be the vertices of a triangle. Show that the area of the triangle is $\frac{1}{2}||\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}||$.

7. The *altitude* of a vertex A in a triangle ABC is the line through A meeting the opposite side OB at a right-angle.

   Let OAB be a triangle and let $\mathbf{a} = \overrightarrow{OA}$, $\mathbf{b} = \overrightarrow{OB}$.



   Let Π be the plane through $\mathbf{0}$, $\mathbf{a}$ and $\mathbf{b}$.

   (a) Show that $\mathbf{v} \in \Pi$ is on the altitude of A if and only if $\mathbf{b} \cdot \mathbf{v} = \mathbf{b} \cdot \mathbf{a}$.

   (b) Give a similar condition for $\mathbf{v} \in \Pi$ to be on the altitude of B.

   (c) Show that $\mathbf{v} \in \Pi$ is on the altitude of O if and only if $(\mathbf{b} - \mathbf{a}) \cdot \mathbf{v} = 0$.

   (d) Deduce that the altitudes of a triangle meet at a point.

8. As in Problem 1.22, let Π be the xz-plane, so $\Pi = \{x\mathbf{i} + z\mathbf{k} : x, z \in \mathbb{R}\}$ and let ℓ be the line through $\mathbf{0}$ and $\mathbf{i} + \sqrt{2/3}\mathbf{j} - \mathbf{k}$.

   (a) Let $0 \leq \gamma < 2\pi$. Let $\theta(\gamma)$ be the angle between ℓ and the line ℓ' passing through $\mathbf{0}$ and $(\cos\gamma, 0, \sin\gamma)$. Show that

$$\cos\theta(\gamma) = \frac{\sqrt{3}}{2}\left|\cos\left(\gamma + \tfrac{\pi}{4}\right)\right|.$$

   (b) Hence find the maximum and minimum values of $\theta(\gamma)$, and the values of γ for which they are achieved.

   (c) Give a geometric interpretation of your answer to (b).

MT182 Matrix Algebra: Sheet 3

Attempt at least questions 1 to 5. Please staple your answers together and put your name and student number on the top sheet. Do not return the problem sheet.

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Feedback. Please write down something from lectures that (a) you enjoyed or found clear, *and/or* (b) is confusing to you, despite some thought.

To be handed in at the 10am lecture on Friday 5th February.

1. Let Π be the plane through $\mathbf{i}+\mathbf{j}$ with normal direction $\mathbf{j}+\mathbf{k}$. Let ℓ be the line through $-\mathbf{j}$ with direction $\mathbf{i}+\mathbf{k}$.

(a) Explain why $\Pi = \{\mathbf{v} \in \mathbb{R}^3 : (\mathbf{j}+\mathbf{k}) \cdot \mathbf{v} = 1\}$.

(b) Show that ℓ meets Π at a unique point and find this point.

(c) Find the angle θ between ℓ and Π.

2. Let Π be the plane in Question 1. Let Π' be the plane through $A = (1, 1, 0)$, $B = (0, 1, 1)$ and $C = (-1, 1, 1)$.

(a) Find $\mathbf{n}' \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$ such that $\Pi' = \{\mathbf{v} \in \mathbb{R}^3 : \mathbf{n}' \cdot \mathbf{v} = \alpha\}$.

(b) Find $\mathbf{a}$, $\mathbf{c} \in \mathbb{R}^3$ such that $\Pi \cap \Pi' = \{\mathbf{a} + \lambda\mathbf{c} : \lambda \in \mathbb{R}\}$. [*Hint: Problem 2.7 is similar.*]

3. Let Π and Π' be the planes through $\mathbf{0}$ with normals $\mathbf{j}$ and $\mathbf{i}+\sqrt{3}\mathbf{j}$. Find the angle between Π and Π'.

4. Let $\mathbf{c} \in \mathbb{R}^3$ and suppose that $||\mathbf{c}|| = 1$. Let ℓ be the line through 0 and $\mathbf{c}$. Let $A \in \mathbb{R}^3$, let $\mathbf{a} = \overrightarrow{OA}$ and let $P \in \ell$ be the closest point in ℓ to A.

(a) Draw a diagram showing O, A, P and the vectors $\mathbf{a}$ and $\mathbf{c}$.

(b) Show that $||\overrightarrow{OP}|| = \mathbf{c} \cdot \overrightarrow{OA}$. [*Hint: consider the right-angled triangle OPA. The quiz question before Theorem 2.12 uses the same idea.*]

(c) Hence show that $\overrightarrow{PA} = \mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{c}$.

(d) Deduce that $||\overrightarrow{PA}||^2 = ||\mathbf{a}||^2 - (\mathbf{c} \cdot \mathbf{a})^2$. [*Hint: use Lemma 1.17.*]

(e) Find $||\overrightarrow{PA}||$ when $\mathbf{c} = \frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$ and $A = (1, 1, 0)$.

5. Find the volume of the parallelepiped formed by the vectors $\mathbf{0}$, $\mathbf{i}+\mathbf{j}$, $\mathbf{j}+\mathbf{k}$ and $\mathbf{i}+\mathbf{j}+\mathbf{k}$.

Bonus questions

6. Find the volume of the cone with base vertices at $\mathbf{0}$, $\mathbf{i}$, $\mathbf{i}+\mathbf{j}$ and apex at $\mathbf{i}+\mathbf{j}+\mathbf{k}$.

7. The dot product of vectors in $\mathbb{R}^n$ is defined by

$$\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \sum_{i=1}^{n} u_i v_i.$$

The Cauchy–Schwarz inequality states that if $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ then

$$|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}||\, ||\mathbf{v}||.$$

(a) Prove the Cauchy–Schwarz inequality when $n = 3$.

(b) Prove that $\left(\sum_{i=1}^{n} x_i\right)^2 \leq n \sum_{i=1}^{n} x_i^2$. [*Hint:* choose $\mathbf{u}$ and $\mathbf{v}$ suitably.]

(c) Prove that if $x, y, z \in \mathbb{R}$ then $xy + yz + zx \leq x^2 + y^2 + z^2$.

(d) Show that if $r, s, t \in \mathbb{R}$, $r, s, t > 0$ and $r + s + t \leq 3$ then $1/r + 1/s + 1/t \geq 3$.

8. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{R}^3$. The 'bac-cab' rule states that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}).$$

Using this identity, prove that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$.

MT182 Matrix Algebra: Sheet 4

Attempt at least questions 1 to 5. Please staple your answers together and put your name and student number on the top sheet. Do not return the problem sheet.

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Feedback. Please write down something from lectures that (a) you enjoyed or found clear, *and/or* (b) is confusing to you, despite some thought.

To be handed in at the 10am lecture on Friday 12th February.

1. The sum of 2×2 matrices and product of a 2×2 matrix by a scalar are defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} a+r & b+s \\ c+t & d+u \end{pmatrix} \quad \text{and} \quad \lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}.$$

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and let $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(a) Compute AB, BA, A^2, B^2, $A+B$, $(A+B)^2$.

(b) True or false? $(A+B)^2 = A^2 + 2AB + B^2$

(c) Express $(A+B)^2$ as a linear combination of A^2, AB, BA and B^2.

Let $C = \begin{pmatrix} 5 & 3 \\ -7 & -4 \end{pmatrix}$.

(d) Find C^2, C^3 and C^6. [*Hint:* $C^6 = (C^3)^2$.]

(e) Hence find C^{2016} and C^{2017}.

2. Let $\theta \in \mathbb{R}$ and let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the function sending $\mathbf{v} \in \mathbb{R}^2$ to $\mathbf{v}$ rotated (anticlockwise) by θ.

(a) By generalizing the solution to Problem 3.8 using complex numbers, prove that f is a linear map and find the 2×2 matrix representing f.

(b) Let A be the matrix representing $f_{\pi/3}$. Write down A and A^3.

3. Prove that $\det AB = (\det A)(\det B)$ for any 2×2 matrices A and B.

4. Which of the following functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ are linear maps? Justify your answers briefly.

(a) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} |x| \\ y \end{pmatrix}$ (b) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ y \end{pmatrix}$ (c) $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y+1 \\ y \end{pmatrix}$

5. (a) For each of the following matrices state whether or not it has an inverse, and find the inverse when it exists.

$$\begin{pmatrix} 5 & 7 \\ 7 & 10 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{7} & \frac{1}{5} \\ \frac{1}{10} & \frac{1}{7} \end{pmatrix}, \quad \begin{pmatrix} 133 & 171 \\ 161 & 207 \end{pmatrix}.$$

(b) Let $f,\, g : \mathbb{R}^2 \to \mathbb{R}^2$ be invertible linear maps represented by the matrices A and B, respectively. Prove that $(gf)^{-1}$ is a linear map and finds its matrix in terms of A^{-1} and B^{-1}. [*Hint:* use Lemma 3.11]

Bonus questions

6. True or false? Only finitely many 2×2 matrices M with integer entries satisfy $M^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Justify your answer.

7. A *permutation matrix* of order $n \in \mathbb{N}$ is an $n \times n$ matrix in which each row and column has a unique non-zero entry, and this entry is 1. For instance

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

is a permutation matrix. Let P and Q be $n \times n$ matrices with non-negative integer entries. Prove that PQ is a permutation matrix if and only if P and Q are permutation matrices.

MT182 Matrix Algebra: Sheet 5

Attempt at least questions 1 to 4. Please staple your answers together and put your name and student number on the top sheet. Do not return the problem sheet.

The lecturer will be happy to discuss any of the questions in office hours.

Feedback. Please write down something from lectures that (a) you enjoyed or found clear, *and/or* (b) is confusing to you, despite some thought.

To be handed in at the 10am lecture on Friday 19th February.

1. Let $A = \begin{pmatrix} 4 & 3 \\ -2 & -1 \end{pmatrix}$.

(a) Find the eigenvalues of A.

(b) For each eigenvalue of A, find a corresponding eigenvector.

(c) Find A^n for all $n \in \mathbb{N}_0$.

(d) What are the eigenvalues of A^2? [*Hint avoiding extra calculation:* if $\mathbf{v}$ is an eigenvector of A with eigenvalue λ, what is $A^2\mathbf{v}$?]

(e) Find $\det A^n$.

2. Let $\lambda \in \mathbb{R}$. Find a 2×2 matrix A with eigenvectors $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$ and eigenvalues 1 and 3. [*Hint:* $A = PDP^{-1}$.]

3. The land of Erewhon is stricken by purple fever and under quarantine. Purple fever is non-fatal, and indeed has no symptoms at all, except to turn the sufferer a vivid shade of purple. Each month 15% of healthy people will become purple and 20% of purple people will recover.

Suppose that there are h_n healthy people and p_n purple people in month n. Assume that $h_1 = 500$ and $p_1 = 200$.

(a) Show that $h_2 = 465$ and find p_2.

(b) Let $A = \begin{pmatrix} \frac{17}{20} & \frac{1}{5} \\ \frac{3}{20} & \frac{4}{5} \end{pmatrix}$. Explain why $\begin{pmatrix} h_{n+1} \\ p_{n+1} \end{pmatrix} = A \begin{pmatrix} h_n \\ p_n \end{pmatrix}$ for all $n \in \mathbb{N}$.

(c) Roughly how many purple people will there be in a year's time?

Please turn over

4. Let $\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix} \in \mathbb{R}^2$ and let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The converse of Lemma 3.23 is:

$$A \text{ is invertible} \implies \begin{pmatrix} a \\ c \end{pmatrix} \text{ and } \begin{pmatrix} b \\ d \end{pmatrix} \text{ are not proportional.}$$

(a) Write down the contrapositive of this proposition.

(b) Prove that the proposition in (a) is true. [*Hint:* show the determinant of A is zero and use Lemma 3.13 to conclude that A is not invertible.]

Bonus questions

5. Prove that a 2×2 matrix A is diagonalizable if and only if it has two non-proportional eigenvectors.

6. Is there a 2×2 matrix A such that $A^2 = \begin{pmatrix} 3 & 4 \\ 4 & 5 \end{pmatrix}$? Justify your answer.

7. Let $0 \leq \theta < 2\pi$ and let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

(a) Find the solutions $\lambda \in \mathbb{C}$ to the equation $\det(\lambda I - A) = 0$.

(b) Deduce that A has an eigenvector if and only if $\theta = 0$ or $\theta = \pi$.

(c) When is A diagonalizable? What is the geometric interpretation of this?

8. (a) Let $\sim$ be the relation defined on $\mathbb{R}^2$ by $\mathbf{u} \sim \mathbf{v} \iff \mathbf{u}$ and $\mathbf{v}$ are proportional. Is $\sim$ an equivalence relation? Justify your answer.

(b) Let $\sim$ be the same relation but now defined on $\mathbb{R}^2 \backslash \{\mathbf{0}\}$. Show that $\sim$ is an equivalence relation and describe the equivalence classes.

MT182 Matrix Algebra: Sheet 6

Attempt at least questions 1 to 5. 5(c) is optional. Please staple your answers together and put your name and student number on the top sheet. Do not return the problem sheet.

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Feedback. Please write down something from lectures that (a) you enjoyed or found clear, *and/or* (b) is confusing to you, despite some thought.

To be handed in at the 10am lecture on Friday 25th February.

1. Recall that the product of an $m \times n$ matrix and an $n \times p$ matrix is an $m \times p$ matrix. Let $A = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and let $B = \begin{pmatrix} 1 & 2 \end{pmatrix}$. So A is a 2×1 matrix and B is a 1×2 matrix.

(a) What are the numbers of rows and columns of AB and BA?

(b) Calculate AB and BA.

(c) Check that $A(BA)B = (AB)^2$.

2. Let $A = \dfrac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$. Let $\mathbf{n} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and let Π be the plane through the origin with normal $\mathbf{n}$.

(a) Show that $A\mathbf{n} = -\mathbf{n}$.

(b) Let $\mathbf{u} = \mathbf{i} - \mathbf{j}$. Find $A\mathbf{u}$.

(c) Find a vector $\mathbf{v}$ such that $\Pi = \{\alpha\mathbf{u} + \beta\mathbf{v} : \alpha, \beta \in \mathbb{R}\}$.

(d) Show that if $\mathbf{w} \in \Pi$ then $A\mathbf{w} = \mathbf{w}$.

(e) What, geometrically, is the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(\mathbf{w}) = A\mathbf{w}$?

3. The *transpose* of an $m \times n$ matrix A is the $n \times m$ matrix A^T defined by $(A^T)_{ij} = A_{ji}$ for $1 \le i \le n$ and $1 \le j \le m$. A matrix A is said to be *symmetric* if $A = A^T$. For example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \text{ is symmetric.}$$

(a) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$. Calculate AA^T, A^TA and $(A^T)^T$.

(b) Show that if A is any $m \times n$ matrix and B is any $n \times p$ matrix then $(AB)^T = B^TA^T$. [*Hint:* find formulae for $(AB)^T_{ij}$ and $(B^TA^T)_{ij}$, as in Proposition 4.9.]

(c) Show that if C is any $m \times n$ matrix then CC^T is symmetric. [*Hint:* use (b).]

4. Let A be an $m \times n$ matrix and let B and C be $n \times p$ matrices. Prove that $A(B+C) = AB + AC$.

5. A matrix M is *stochastic* if its entries are non-negative and the sum of the entries in each row is 1.

(a) Let M be an $n \times n$ matrix with non-negative entries. Let $\mathbf{v} \in \mathbb{R}^n$ be defined by $v_i = 1$ for $1 \leq i \leq n$. Show that M is stochastic if and only if $M\mathbf{v} = \mathbf{v}$.

(b) Let A and B be $n \times n$ matrices. Show that if A and B are stochastic then AB is stochastic.

(c) (Optional.) Does the converse hold in (b)?

Bonus questions

6. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix. Let $P(x) = x^2 - (a+d)x + (ad - bc)$.

(a) Show that the eigenvalues of A are the roots of P.

(b) Show that the sum of the eigenvalues of A is $a+d$ and their product is $\det A$.

(c) Show that $P(A) = 0$. (A special case of the Cayley–Hamilton Theorem.)

7. Show that

$$\{AB - BA : A,\, B \text{ are } 2 \times 2 \text{ matrices}\} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

Hence, or otherwise, show that if A, B and C are 2×2 matrices then $(AB - BA)^2$ commutes with C.

8. The Fibonacci numbers F_n are defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

(a) Prove by induction that

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix}$$

for all $n \in \mathbb{N}$.

(b) By diagonalizing $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, or otherwise, find a formula for F_n.

(c) Prove that $F_{n+1}F_{n-1} = F_n^2 + (-1)^n$ for all $n \in \mathbb{N}$.

MT182 Matrix Algebra: Sheet 7

Attempt at least questions 1 to 5. Please staple your answers together and put your name and student number on the top sheet. Do not return the problem sheet.

The lecturer will be happy to discuss any of the questions in office hours.

Feedback. Please write down something from lectures that (a) you enjoyed or found clear, *and/or* (b) is confusing to you, despite some thought.

To be handed in at the 10am lecture on Friday 4th March.

1. Find the unique real number α such that following system of equations has a solution.

$$\begin{aligned} x - y + 2z &= 0 \\ 2x + y + 7z &= 1 \\ y + z &= \alpha \end{aligned}$$

Find *all* solutions in this case.

2. Let A be an $m \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^n$. Let $(A \,|\, \mathbf{b})$ denote the *augmented* $m \times (n+1)$ matrix obtained from A by appending $\mathbf{b}$ as a new column. For example, if

$$A = \begin{pmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}, \text{ then } (A \,|\, \mathbf{b}) = \begin{pmatrix} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{pmatrix}.$$

Suppose that $(A' \,|\, \mathbf{b}')$ is obtained from $(A \,|\, \mathbf{b})$ by a sequence of elementary row operations. Show that

$$\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\} = \{\mathbf{x} \in \mathbb{R}^n : A'\mathbf{x} = \mathbf{b}'\}.$$

[*Hint:* adapt the proof of Corollary 5.5.]

3. Let $A = \begin{pmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -1 & 3 \\ -1 & 1 & -1 & 0 \end{pmatrix}$ and let $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}$.

(a) Use row operations to put $(A \,|\, \mathbf{b})$ in row-reduced echelon form. Make it clear what operations you perform.

(b) Hence, using Question 2, find all solutions $\mathbf{x} \in \mathbb{R}^4$ to the equation $A\mathbf{x} = \mathbf{b}$.

4. Let $A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$ be a general 3×3 matrix and let $Q = \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) Calculate QA. How are the rows of QA related to the rows of A?

(b) Calculate AQ. How are the columns of AQ related to the columns of A?

(c) State a generalization of (b) for the case when A is an $m \times n$ matrix. Clearly define the matrix replacing Q.

5. For which $d \in \mathbb{N}_0$ is there a 3×6 matrix A such that

$$\{\mathbf{x} \in \mathbb{R}^6 : A\mathbf{x} = \mathbf{0}\}$$

is d-dimensional? Justify your answer.

[*Hint:* you can interpret d-dimensional to mean that the solution set has d independent parameters. For example, the solution set to Example 5.2 (in printed notes) is the 2-dimensional plane

$$\left\{ \begin{pmatrix} -2x_2 + x_4 \\ x_2 \\ -x_4 \\ x_4 \end{pmatrix} : x_2, x_4 \in \mathbb{R} \right\},$$

and the solution set in Example 5.2′ is the 1-dimensional line

$$\left\{ \begin{pmatrix} -z \\ -z \\ z \end{pmatrix} : z \in \mathbb{R} \right\}.]$$

Bonus questions

6. Let A be an $m \times n$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$. Show that the equation

$$A\mathbf{x} = \mathbf{b}$$

has a solution $\mathbf{x} \in \mathbb{R}^n$ if and only if there exist scalars $\alpha_1, \ldots, \alpha_n$ such that

$$\alpha_1\mathbf{a}_1 + \cdots + \alpha_n\mathbf{a}_n = \mathbf{b}.$$

7. Prove that the row-reduced echelon form of a matrix A is unique. [*Hint:* there is a proof by induction on the number of columns of A.]

8. Write a computer program, in the language of your choice (MATHEMATICA is one possibility), to put a matrix in row-reduced echelon form.

MT182 Matrix Algebra: Sheet 8

Attempt at least questions 1 to 5. Please staple your answers together and put your name and student number on the top sheet. Do not return the problem sheet.

The lecturer will be happy to discuss any of the questions in office hours.

Feedback. Please write down something from lectures that (a) you enjoyed or found clear, *and/or* (b) is confusing to you, despite some thought.

To be handed in at the 10am lecture on Friday 11th March.

1. Find the inverse or state that no inverse exists for each of the following matrices:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 5 & 6 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

2. Find all solutions to the equations

$$\begin{aligned} x + y + z + w &= 2 \\ x - y + z - w &= 2 \\ x + 2y + 3z + 4w &= 2 \end{aligned}$$

For which $\alpha \in \mathbb{R}$ is there a solution with $x + y = \alpha$?

3. (a) Show that if A is 3×3 matrix then $\det A = \det A^T$, where A^T is the transpose of A.

(b) Hence, or otherwise, show that

$$\det A = A_{11} \det \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} - A_{12} \det \begin{pmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{pmatrix} + A_{13} \det \begin{pmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{pmatrix}.$$

[*Hint:* your answers must deal with a general 3×3 matrix. You may use either part of Lemma 6.2.]

4. Let σ be the permutation of $\{1, 2, 3, 4, 5, 6\}$ defined by $\sigma(1) = 6$, $\sigma(2) = 1$, $\sigma(3) = 5$, $\sigma(4) = 4$, $\sigma(5) = 3$, $\sigma(6) = 2$. Let τ be the transposition $(2, 3)$.

(a) Write σ and τ in two line form.

(b) Write σ in disjoint cycle form.

(c) Write $\sigma\tau$ in disjoint cycle form.

(d) Find all $m \in \mathbb{Z}$ such that σ^m is the identity permutation.

5. True or false? Justify your answers in each case.

(a) If A is a 3×3 matrix then A is invertible if and only if its row-reduced echelon form is the 3×3 identity matrix.

(b) If A and B are invertible square matrices then $A + B$ is invertible.

(c) Let A be an $m \times n$ matrix and let $\mathbf{b} \in \mathbb{R}^m$. If the equation $A\mathbf{x} = \mathbf{b}$ has two solutions in $\mathbb{R}^n$ then it has infinitely many solutions.

(d) If A is a $n \times n$ matrix such that $A^n = 0$ then $I + A$ is invertible.

Bonus questions

6. Prove that if A, B and C are square matrices such that $BA = AB = CA = AC = I$ then $B = C$. (Thus the inverse of a matrix is unique, when it exists.)

7. Given a polynomial $f(X_1, \ldots, X_n)$ in indeterminates $X_1, \ldots, X_n$ and a permutation σ of $\{1, 2, \ldots, n\}$, let $\sigma \cdot f$ be the polynomial defined by

$$(\sigma \cdot f)(X_1, \ldots, X_n) = f(X_{\sigma(1)}, \ldots, X_{\sigma(n)}).$$

For example, if $n = 4$ and σ is the double transposition $(1, 2)(3, 4)$ then $\sigma \cdot X_1 X_3 = X_2 X_4$ and $\sigma \cdot (X_1 - X_2)(X_3 - X_4) = (X_1 - X_2)(X_3 - X_4)$.

Let $g(X_1, \ldots, X_n) = \prod_{i<j}(X_i - X_j)$. For example, if $n = 3$ then $g(X_1, X_2, X_3) = (X_1 - X_2)(X_1 - X_3)(X_2 - X_3)$.

(a) Show that if $1 \leq a < b \leq n$ then $(a, b) \cdot g = -g$.

(b) Show that if σ can be expressed as a composition of k transpositions then $\sigma \cdot g = (-1)^k g$.

(c) Hence prove that the sign of a permutation is well-defined.

8. Say that a matrix is *elementary* if it is one of the matrices in Lemma 5.4 corresponding to the three elementary row operations.

Let A be an $n \times n$ matrix and let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the corresponding linear map, defined by $f(\mathbf{x}) = A\mathbf{x}$ for $\mathbf{x} \in \mathbb{R}^n$.

Prove that the following are equivalent:

(i) A is invertible;

(ii) f is injective;

(iii) A is equal to a product of elementary matrices.

[*Hint:* see the optional extras for §5 in the printed notes for a related result.]

MT182 Matrix Algebra: Sheet 9

Attempt at least questions 1 to 5. Question 2(d) is optional. Please staple your answers together and put your name and student number on the top sheet. Do not return the problem sheet.

The lecturer will be happy to discuss any of the questions in office hours.

Feedback. Please write down something from lectures that (a) you enjoyed or found clear, *and/or* (b) is confusing to you, despite some thought.

To be handed in at the 10am lecture on Friday 18th March.

1. Let $\sigma = (1, 4, 2)(3, 6, 5, 7)$ and let $\tau = (2, 3, 4)$.

(a) Write $\tau\sigma$ in two line form.

(b) Write $\tau\sigma\tau^{-1}$ as a composition of disjoint cycles.

(c) State the values of $\mathrm{sgn}(\sigma), \mathrm{sgn}(\tau), \mathrm{sgn}(\sigma^{-1})$ and $\mathrm{sgn}(\tau\sigma)$. Are your answers consistent with Lemma 6.7?

2. (a) Let $(a_1, a_2, \ldots, a_k)$ be a k-cycle with $a_1, \ldots, a_k \in \{1, 2, \ldots, n\}$. Show that if τ is any permutation of $\{1, 2, \ldots, n\}$ then

$$\tau(a_1, a_2, \ldots, a_k)\tau^{-1} = \big(\tau(a_1), \tau(a_2), \ldots, \tau(a_k)\big).$$

(b) Is there a permutation τ of $\{1, 2, 3, 4, 5, 6\}$ such that $\tau(1, 2, 3)(4, 5, 6)\tau^{-1} = (1, 3, 5)(2, 4, 6)$? Justify your answer.

(c) Is there a permutation τ of $\{1, 2, 3, 4, 5, 6\}$ such that $\tau(1, 2, 3)(4, 5, 6)\tau^{-1} = (1, 2, 3, 4, 5, 6)$? Justify your answer.

[*Hint:* $\tau(1, 2, 3)(4, 5, 6)\tau^{-1} = \tau(1, 2, 3)\tau^{-1}\tau(4, 5, 6)\tau^{-1}$.]

(d) (Optional.) Does your answer to (b) change if we require $\mathrm{sgn}(\tau) = 1$?

3. Which property of determinants implies that

$$\det\begin{pmatrix} 1 & \alpha & \beta - 1 \\ 0 & \alpha & \beta \\ \gamma & \delta & 0 \end{pmatrix} = \det\begin{pmatrix} 0 & \alpha & \beta \\ 0 & \alpha & \beta \\ \gamma & \delta & 0 \end{pmatrix} + \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ \gamma & \delta & 0 \end{pmatrix} + \det\begin{pmatrix} 0 & 0 & -1 \\ 0 & \alpha & \beta \\ \gamma & \delta & 0 \end{pmatrix}?$$

Using the equation above, or otherwise, evaluate the left-hand side.

4. Show that

$$\det\begin{pmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{pmatrix} = (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha).$$

5. Two sliding block puzzles are shown below with initial position on the left and target position on the right. Which are soluble? Justify your answers.

(a)

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	$\square$

15	14	13	12
11	10	9	8
7	6	5	4
3	2	1	$\square$

(b)

D	E	A	D
P	I	G	S
F	L	Y	$\square$
T	O	W	N

D	E	A	D
P	I	G	S
W	O	N	T
F	L	Y	$\square$

[*Hint:* you may assume that any permutation with sign 1 fixing $\square$ is a composition of a suitable sequence of slides. Please think extra carefully about (b).

Bonus questions

6. Show that

$$\det\begin{pmatrix} 0 & \alpha & \beta & \gamma \\ \gamma & 0 & \alpha & \beta \\ \beta & \gamma & 0 & \alpha \\ \alpha & \beta & \gamma & 0 \end{pmatrix} = (\alpha+\beta+\gamma)(\beta-\alpha-\gamma)(\alpha^2+\beta^2+\gamma^2-2\alpha\gamma).$$

7. Let $n \geq 3$ be a natural number. Let Alt_n be the set of all permutations σ of $\{1,2,\ldots,n\}$ such that $\mathrm{sgn}(\sigma) = 1$.

(a) Let τ be a permutation of $\{1,2,\ldots,n\}$. Show that $\tau \in \mathrm{Alt}_n$ if and only if $\tau(1,2) \notin \mathrm{Alt}_n$.

(b) Hence find $|\mathrm{Alt}_n|$.

(c) Show that if σ, $\tau \in \mathrm{Alt}_n$ then $\sigma\tau \in \mathrm{Alt}_n$.

(d) Show that any element of Alt_n can be expressed as a composition of 3-cycles.

8. Let G be the set of all permutations of $\{1,2,3,4,5,\square\}$ obtainable by a sequence of slides on the 2×3 sliding block puzzle shown below.

1	2	3
4	5	$\square$

(a) Show that $(2,3,5) \in G$.

(b) Show that G contains a 3-cycle moving both 1 and 4 and fixing $\square$.

(c) Show that G contains every permutation σ of $\{1,2,3,4,5,\square\}$ such that $\mathrm{sgn}(\sigma) = 1$.

MT182 Matrix Algebra: Sheet 10

Attempt at least questions 1 to 5. This problem sheet need not be handed in, but you are welcome to discuss the questions in workshops or office hours. You are also encouraged to ask revision questions in workshops in the final week of term.

A mock exam and examination guidance are available from Moodle.

1. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Compute $\det A$, $\operatorname{adj} A$, $A(\operatorname{adj} A)$ and $(\operatorname{adj} A)A$.

2. Let $A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$. Let $C = A(\operatorname{adj} A)$.

(a) Show that $C_{11} = \det A$. [*Hint:* use Proposition 7.9.]

(b) Show that

$$C_{12} = \det \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{11} & A_{12} & A_{13} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

Hence show that $C_{12} = 0$.

(c) Show similarly that $C_{13} = 0$.

Remark: This shows that C and $(\det A)I$ are equal in their first rows. To complete the proof that $C = (\det A)I$ one argues similarly, expanding on rows 2 and 3: see Question 7 for the relevant generalization of Proposition 7.9.

3. Find the eigenvalues of

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -3 & 4 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

and find an eigenvector corresponding to each eigenvalue. Hence find an invertible 3×3 matrix P such that $P^{-1}AP$ is diagonal.

4. Let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 4 \\ 0 \\ 2 \\ 2 \end{pmatrix}$, $\mathbf{z} = \begin{pmatrix} 4 \\ 2 \\ 3 \\ -4 \end{pmatrix}$.

(a) Find scalars α, β, γ, δ, not all equal to zero, such that $\alpha\mathbf{u}+\beta\mathbf{v}+\gamma\mathbf{w}+\delta\mathbf{z} = \mathbf{0}$.

(b) Find a basis for the subspace of $\mathbb{R}^4$ spanned by $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{z}$. [*Hint:* Problem 8.4 is similar.]

5. Let V be the set of all 2×2 matrices. You may assume that V is a vector space.

(a) Show that $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is a basis for V. What is $\dim V$?

(b) Which of the following subsets of V are subspaces? Justify your answers.

(i) $W_1 = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} : a, b \in \mathbb{R} \right\}$.

(ii) $W_2 = \{A \in V : A_{11} \neq 0\}$.

(iii) $W_3 = \{A \in V : A = A^T\}$.

Bonus questions

6. Let A be an $n \times n$ matrix and let σ be a permutation of $\{1, 2, \ldots, n\}$ such that $\sigma(1) = \ell$. Let $\tau : \{1, 2, \ldots, n-1\} \to \{1, 2, \ldots, n-1\}$ be the permutation defined by

$$\tau(i) = \begin{cases} \sigma(i+1) & \text{if } \sigma(i+1) < \ell \\ \sigma(i+1) - 1 & \text{if } \sigma(i+1) > \ell. \end{cases}$$

(Note that $\sigma(i+1) \neq \ell$, since $\sigma(1) = \ell$.)

(a) Show that $A_{1\ell}A_{2\sigma(2)} \ldots A_{n\sigma(n)} = A_{1\ell}M(1,\ell)_{1\tau(1)} \ldots M(1,\ell)_{(n-1)\tau(n-1)}$.

(b) Show that $\operatorname{sgn}(\tau) = (-1)^{\ell+1} \operatorname{sgn}(\sigma)$.

Deduce that

$$\det A = A_{11} \det M(1,1) - A_{12} \det M(1,2) + \cdots + (-1)^{n+1} A_{1n} \det M(1,n)$$

as required for Proposition 7.9.

7. Let A be an $n \times n$ matrix. Show that if $1 \leq k \leq n$ then

$$(-1)^{k+1} \det A = A_{k1} \det M(k,1) - A_{k2} \det M(k,2) + \cdots + (-1)^{n+1} A_{kn} \det M(k,n).$$

[*Hint:* use a row operation to move row k to the top and rows 1, …, $k-1$ each down one position, then apply Proposition 7.9.]

8. Let $A = \begin{pmatrix} 0 & 1 & \ldots & 1 \\ 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 0 \end{pmatrix}$ where the matrix is $n \times n$. (Compare Example 7.15′.)

(a) Write down an eigenvector of A with eigenvalue $n-1$.

(b) Write down $n-1$ linearly independent eigenvectors of A with eigenvalue -1.

(c) Hence find an $n \times n$ matrix P such that $P^{-1}AP$ is diagonal.

(d) Let d_+ and d_- be the numbers of derangements of $\{1, 2, \ldots, n\}$ that have sign $+1$ and -1, respectively. Using (c), find a formula for $d_+ - d_-$.