A NEW MODULAR PLETHYSTIC $\mathrm{SL}_2(\mathbb{F})$-ISOMORPHISM $\mathrm{Sym}^{N-1}E \otimes \bigwedge^{N+1}\mathrm{Sym}^{d+1}E \cong \Delta^{(2,1^{N-1})}\mathrm{Sym}^d E$

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Abstract. Let $\mathbb{F}$ be a field and let E be the natural representation of $\mathrm{SL}_2(\mathbb{F})$. Given a vector space V, let $\Delta^{(2,1^{N-1})}V$ be the kernel of the multiplication map $\bigwedge^N V \otimes V \to \bigwedge^{N+1} V$. We construct an explicit $\mathrm{SL}_2(\mathbb{F})$-isomorphism $\mathrm{Sym}^{N-1}E \otimes \bigwedge^{N+1}\mathrm{Sym}^{d+1}E \cong \Delta^{(2,1^{N-1})}\mathrm{Sym}^d E$. This $\mathrm{SL}_2(\mathbb{F})$-isomorphism is a modular lift of the q-binomial identity $q^{\frac{N(N-1)}{2}}[N]_q\left[{d+1 \atop N+1}\right]_q = s_{(2,1^{N-1})}(1,q,\ldots,q^d)$, where $s_{(2,1^{N-1})}$ is the Schur function for the partition $(2,1^{N-1})$. This identity, which follows from our main theorem, implies the existence of an isomorphism when $\mathbb{F}$ is the field of complex numbers but it is notable, and not typical of the general case, that there is an explicit isomorphism defined in a uniform way for any field.

1. Introduction

Let $\mathbb{F}$ be an arbitrary field and let E be the natural 2-dimensional representation of the special linear group $\mathrm{SL}_2(\mathbb{F})$. Let Δ^λ denote the Schur functor canonically labelled by the partition λ. Working over the field of complex numbers there is a rich theory of plethystic isomorphisms between the representations $\Delta^\lambda\,\mathrm{Sym}^d E$. These include Hermite reciprocity and the Wronskian isomorphism; we refer the reader to [PW21] for a comprehensive account and references to earlier results. In [McDW] it was shown that both these classical isomorphisms hold over an arbitrary field, *provided that suitable dualities are introduced.* The modular version of Hermite reciprocity is $\mathrm{Sym}_M\,\mathrm{Sym}^d E \cong \mathrm{Sym}^d\,\mathrm{Sym}_M E$, where, given a $\mathrm{SL}_2(\mathbb{F})$-representation V, $\mathrm{Sym}^r V$ is the symmetric power defined as a quotient of $V^{\otimes r}$ and $\mathrm{Sym}_r V$ is its dual defined as the subspace of invariant tensors in $V^{\otimes r}$. The modular Wronskian isomorphism is $\mathrm{Sym}_M\,\mathrm{Sym}^d E \cong \bigwedge^M \mathrm{Sym}^{d+M-1}E$. The purpose of this article is to add to the collection of such modular plethystic isomorphisms by proving the following theorem. The version of the Schur functor $\Delta^{(2,1^{N-1})}$ we require is defined in §1.1 immediately below.

1991 *Mathematics Subject Classification.* Primary: 20C20, Secondary: 05E05, 05E10, 17B10, 20G05.

Theorem 1.1. *Let $N \in \mathbb{N}$ and let $d \in \mathbb{N}_0$. The map φ defined in Definition 1.5 is an isomorphism of* $\mathrm{SL}_2(\mathbb{F})$*-representations*

$$\mathrm{Sym}^{N-1}E \otimes \bigwedge^{N+1} \mathrm{Sym}^{d+1}E \cong \Delta^{(2,1^{N-1})}\,\mathrm{Sym}^d E.$$

This theorem is notable as the first explicit example of a modular plethystic isomorphism involving a Schur functor for a partition that is not one-row or one-column, and also for the unexpected tensor factorisation it exhibits.

This isomorphism is a modular lift of the q-binomial identity

$$q^{\frac{N(N-1)}{2}}[N]_q \begin{bmatrix} d+2 \\ N+1 \end{bmatrix}_q = s_{(2,1^{N-1})}(1, q, \ldots, q^d) \tag{1.1}$$

where $s_{(2,1^{N-1})}$ denotes the Schur function labelled by the partition $(2, 1^{N-1})$. We structure our proof so that we can obtain (1.1) as a fairly routine corollary of Theorem 1.1: see Corollary 3.2, where we also give combinatorial interpretations of each side. As shown in [McDW, Theorem 1.6] there exist representations of the form $\Delta^\lambda \,\mathrm{Sym}^d E$ that have equal q-characters in the sense of (1.1), and so are isomorphic over $\mathbb{C}$, but fail to be isomorphic over arbitrary fields $\mathbb{F}$, even after considering all possible dualities. Indeed, the authors believe this is the generic case. This adds to be interest and importance of Theorem 1.1. We finish with Corollary 3.1, which lifts the isomorphism in Theorem 1.1 to an isomorphism of representations of $\mathrm{GL}_2(\mathbb{F})$, and Conjecture 3.3 on a conjectured more general isomorphism.

1.1. **Preliminaries.** Fix a basis X, Y of the $\mathbb{F}$-vector space E. For each $r \in \mathbb{N}_0$, the symmetric power $\mathrm{Sym}^c E$ has as a basis the monomials $X^{c-i}Y^i$ for $0 \le i \le c$.

Schur functor. It will be convenient to define the Schur functor $\Delta^{(2,1^{N-1})}$ on a vector space V by

$$\Delta^{(2,1^{N-1})}\, V = \ker \mu_N : \bigwedge^{N} V \otimes V \to \bigwedge^{N+1} V \tag{1.2}$$

where μ_N is the multiplication map $v_1 \wedge \cdots \wedge v_N \otimes w \mapsto v_1 \wedge \cdots \wedge v_N \wedge w$. Thus $\Delta^{(2,1^{N-1})}\, V$ is a subspace of $\bigwedge^N V \otimes V$. Since μ_N is a homomorphism of representations of $\mathrm{GL}(V)$, for any fixed group G, $\Delta^{(2,1^{N-1})}$ is a functor on the category of $\mathbb{F}$-representations of G.

Multi-indices. For c, $r \in \mathbb{N}_0$, let $I^{(c)}(r)$ denote the set $\{0, 1, \ldots, c\}^r$. We say that the elements of $I^{(c)}(r)$ are *multi-indices*. We define the *sum* of a multi-index $\mathbf{i}$ by $|\mathbf{i}| = \sum_{i=1}^r i_\alpha$. Given $\mathbf{i} \in I^{(c)}(r)$ we define

$$F_\wedge^{(c)}(\mathbf{i}) = X^{c-i_1}Y^{i_1} \wedge \cdots \wedge X^{c-i_r}Y^{i_r}.$$

Thus $\bigwedge^r \mathrm{Sym}^c E$ has as a basis all $F_\wedge^{(c)}(\mathbf{i})$ for strictly increasing $\mathbf{i} \in I^{(c)}(r)$.

Definition 1.2. Let $d \in \mathbb{N}_0$. Let $\mathbf{i} \in I^{(d)}(N)$ and $j \in \{0, 1, \dots, d\}$. We define $F(\mathbf{i}, j) \in \bigwedge^N \operatorname{Sym}^d E \otimes \operatorname{Sym}^d E$ by

$$F(\mathbf{i}, j) = F_{\wedge}^{(d)}(\mathbf{i}) \otimes X^{d-j} Y^j. \tag{1.3}$$

We define

$$F_{\Delta}(\mathbf{i}, j) = \begin{cases} F(\mathbf{i}, j) & \text{if } i_1 = j \\ F(\mathbf{i}, j) + F\big((j, i_2, \dots, i_N), i_1\big) & \text{if } i_1 \neq j. \end{cases} \tag{1.4}$$

Given $w \in \mathbb{N}_0$ such that $|\mathbf{i}| \leq w \leq |\mathbf{i}| + d$, we define

$$F^w(\mathbf{i}) = F(\mathbf{i}, w - |\mathbf{i}|). \tag{1.5}$$

It is immediate from (1.5) and then (1.3) that

$$F^w(\mathbf{i}) = F_{\wedge}^{(d)}(\mathbf{i}) \otimes X^{d-(w-|\mathbf{i}|)} Y^{w-|\mathbf{i}|}. \tag{1.6}$$

We say that $(\mathbf{i}, j)$ is *semistandard* if $j \geq i_1$. Observe that $(\mathbf{i}, j)$ is semistandard if and only if the $(2, 1^{N-1})$-tableau $t_{(\mathbf{i},j)}$ shown in the margin having entry i_α in box $(\alpha, 1)$ and entry j in box $(1, 2)$ is semistandard in the usual sense. Whenever we use the notation of Definition 1.2, the value of d will be clear from context. To give an example we take $d = 3$. Then, omitting some parentheses for readability, we have $F\big((1, 2, 3), 0\big) = F^6(1, 2, 3) = X^2Y \wedge XY^2 \wedge Y^3 \otimes X^3$ and

$$t_{(\mathbf{i},j)} = \begin{array}{|c|c|} \hline i_1 & j \\ \hline i_2 & \multicolumn{1}{c}{} \\ \cline{1-1} \vdots & \multicolumn{1}{c}{} \\ \cline{1-1} i_N & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

$$F_{\Delta}\big((1, 2, 3), 0\big) = X^2Y \wedge XY^2 \wedge Y^3 \otimes X^3 + X^3 \wedge XY^2 \wedge Y^3 \otimes X^2Y.$$

Here $\big((1, 2, 3), 0\big)$ is not semistandard in the sense of Definition 1.2, but the equation above shows that $F_{\Delta}\big((1, 2, 3), 0\big) = F_{\Delta}\big((0, 2, 3), 1\big)$, and $\big((0, 2, 3), 1\big)$ *is* semistandard. The convenience of having the two equivalent notations $F^w(\mathbf{i})$ and $F(\mathbf{i}, w - |\mathbf{i}|)$ for the canonical basis elements of $\bigwedge^N \operatorname{Sym}^d E \otimes \operatorname{Sym}^d E$, will be seen many times below.

Lemma 1.3. *There are* $N\binom{d+2}{N+1}$ *semistandard Young tableaux* $(\mathbf{i}, j)$ *of shape* $(2, 1^{N-1})$ *and entries in* $\{0, 1, \dots, N\}$.

Proof. Let S_α be the set of all Young tableaux $t_{(\mathbf{i},j)}$ such that $i_\alpha \leq j < i_{\alpha+1}$. It is clear that the set of semistandard Young tableaux of shape $(2, 1^{N-1})$ is partitioned into the N disjoint subsets $S_1, \dots, S_N$. We claim that each S_α has the same cardinality $\binom{d+2}{N+1}$. To see this, we define a bijection from S_α to the set of strictly increasing multi-indices in $I_{N+1}^{(d+1)}$ by

$$(\mathbf{i}, j) \mapsto (i_1, \dots, i_\alpha, j + 1, i_{\alpha+1} + 1, i_{\alpha+2} + 1, \dots, i_N + 1)$$

The inverse of this map is easily shown to be

$$(k_1, \dots, k_{N+1}) \mapsto \big((k_1, \dots, k_\alpha, k_{\alpha+2} - 1, \dots, k_{N+1} - 1), k_{\alpha+1} - 1\big). \qquad \square$$

Semistandard basis. It is clear that $\bigwedge^N \mathrm{Sym}^d E \otimes \mathrm{Sym}^d E$ has as a canonical basis all $F(\mathbf{i}, j)$ for $\mathbf{i} \in I^{(d)}(N)$ and $j \in \{0, 1, \ldots, d\}$. Observe that if $\mathbf{i} \in I^{(d)}(N)$ and $0 \leq j \leq d$ then

$$\mu_N F\big((j, i_2, \ldots, i_N), i_1\big) = X^{d-j} Y^j \wedge \bigwedge_{\alpha=2}^{N} X^{d-i_\alpha} Y^{i_\alpha} \wedge X^{d-i_1} Y^{i_1}.$$

Up to a swap of the first and final factors, the right-hand side agrees with $\mu_N\big(F(\mathbf{i}, j)\big)$. Hence, by the definition of F_Δ in (1.4), we have $F_\Delta(\mathbf{i}, j) \in \ker \mu_N = \Delta^{(2,1^{N-1})} \mathrm{Sym}^d E$.

Lemma 1.4. *The vector space $\Delta^{(2,1^{N-1})} \mathrm{Sym}^d E$ has dimension $N\binom{d+2}{N+1}$ and a basis*

$$\big\{F_\Delta(\mathbf{i}, j) : \mathbf{i} \in I^{(d)}(N), 0 \leq j \leq d, (\mathbf{i}, j) \text{ semistandard}\big\}.$$

Proof. By considering the maximum j term in a linear relation between the $F_\Delta(\mathbf{i}, j)$ for semistandard $(\mathbf{i}, j)$, one easily sees that these elements are linearly independent. We give a dimension counting argument to show that they span $\Delta^{(2,1^{N-1})} \mathrm{Sym}^d E$. By Lemma 1.3, it suffices to show that $\dim \Delta^{(2,1^{N-1})} \mathrm{Sym}^d E = N\binom{d+2}{N+1}$. This follows from the rank-nullity formula applied to (1.2):

$$\begin{aligned}
\dim \ker \mu_N &= \binom{d+1}{N}(d+1) - \binom{d+1}{N+1} \\
&= \binom{d+1}{N}(d+2) - \left(\binom{d+1}{N} + \binom{d+1}{N+1}\right) \\
&= \binom{d+2}{N+1}(N+1) - \binom{d+2}{N+1} \\
&= N\binom{d+2}{N+1}.
\end{aligned}$$

□

1.2. **Definition of φ.** Given $0 \leq j < k$, set $[j, k) = \{j, j+1, \ldots, k-1\}$. Given a strictly increasing multi-index $\mathbf{k} \in I^{(d+1)}(N+1)$, we define

$$\mathcal{B}(\mathbf{k}) = [k_1, k_2) \times [k_2, k_3) \times \cdots \times [k_N, k_{N+1}) \subseteq I^{(d)}(N). \tag{1.7}$$

For example if $d = 5$, we have

$$\mathcal{B}(0, 2, 3, 6) = [0, 2) \times [2, 3) \times [3, 6) = \{0, 1\} \times \{2\} \times \{3, 4, 5\} \subseteq I^{(5)}(N).$$

Definition 1.5. Fix $d \in \mathbb{N}_0$ and $N \in \mathbb{N}$. We define

$$\varphi : \mathrm{Sym}^{N-1} E \otimes \bigwedge^{N+1} \mathrm{Sym}^{d+1} E \to \bigwedge^{N} \mathrm{Sym}^d E \otimes \mathrm{Sym}^d E$$

by

$$\varphi\big(X^{N-1-s} Y^s \otimes F_\wedge^{(d+1)}(\mathbf{k})\big) = \sum_{\mathbf{i} \in \mathcal{B}(\mathbf{k})} F^{s+|\mathbf{k}|-N}(\mathbf{i})$$

where $0 \leq s \leq N-1$ and $\mathbf{k} \in I^{(d+1)}_{N+1}$ is strictly increasing.

It is immediate from the definition of $F^w(\mathbf{i})$ in (1.5) that

$$\varphi\bigl(X^{N-1-s}Y^s \otimes F_\wedge^{(d+1)}(\mathbf{k})\bigr) = \sum_{\mathbf{i}\in\mathcal{B}(\mathbf{k})} F\bigl(\mathbf{i}, s+|\mathbf{k}|-N-|\mathbf{i}|\bigr) \tag{1.8}$$

where we remind the reader that, by (1.3), the summand on the right hand side is $F_\wedge^{(d)}(\mathbf{i}) \otimes X^{d-(s+|\mathbf{k}|-N-|\mathbf{i}|)}Y^{s+|\mathbf{k}|-N-|\mathbf{i}|}$, or written out in full,

$$X^{d-i_1}Y^{i_1} \wedge \cdots \wedge X^{d-i_N}Y^{i_N} \otimes X^{d-(s+|\mathbf{k}|-N-|\mathbf{i}|)}Y^{s+|\mathbf{k}|-N-|\mathbf{i}|}.$$

As motivation and an aide-memoire, we note that a canonical basis element of Y-degree $s+|\mathbf{k}|$ maps under φ to a sum of canonical basis elements each of Y-degree $s+|\mathbf{k}|-N$. It is not obvious that φ has image in the subrepresentation $\Delta^{(2,1^{N-1})}\operatorname{Sym}^d E$ of $\bigwedge^N V \otimes V$: we give a short proof of this fact in Lemma 2.1.

Example 1.6.
(a) By (1.8), the canonical basis element

$$X^{N-1} \otimes F_\wedge^{(d+1)}(0,1,\ldots,N) = X^{N-1} \otimes X^{d+1} \wedge X^dY \wedge \cdots \wedge X^{d-N+1}Y^N$$

in $\operatorname{Sym}^{N-1}E \otimes \bigwedge^{N+1}\operatorname{Sym}^{d+1}E$ of minimal Y-degree $0+1+\cdots+N$ maps under φ to the canonical basis element

$$F\bigl((0,1,\ldots,N-1),0\bigr) = X^d \wedge X^{d-1}Y \wedge \cdots \wedge X^{d-N+1}Y^{N-1} \otimes X^d$$

in $\Delta^{(2,1^{N-1})}\operatorname{Sym}^d E$ of minimal Y-degree $0+1+\cdots+(N-1)$. Working over $\mathbb{C}$, these vectors are highest weight for the action of the Lie algebra generator e (which may be thought of as $X\frac{\mathrm{d}}{\mathrm{d}Y}$) in (2.2).

(b) More generally the image of $X^{N-1-s}Y^s \otimes F_\wedge^{(d+1)}(i,i+1,\ldots,i+N)$ is $F\bigl((i,i+1,\ldots,i+N-1),s+i\bigr)$. Note that since $s\in\{0,\ldots,N-1\}$, it follows from (1.2) that this image is in $\Delta^{(2,1^{N-1})}\operatorname{Sym}^d E$.

(c) The image of a canonical basis element of $\operatorname{Sym}^{N-1}E \otimes \bigwedge^{N+1}\operatorname{Sym}^{d+1}E$ typically has many summands. For instance take $N=3$ and $d=5$. Then

$$\begin{aligned}
&\varphi(X^3Y \otimes X^6 \wedge X^3Y^2 \wedge X^2Y^3 \wedge Y^6)\\
&= \sum_{\mathbf{i}\in[0,2)\times[2,3)\times[3,6)} F^{1+|(0,2,3,6)|-3}(\mathbf{i})\\
&= F^9(0,2,3)+F^9(1,2,3)+F^9(0,2,4)+F^9(1,2,4)+F^9(0,2,5)+F^9(1,2,5)\\
&= F\bigl((0,2,3),4\bigr)+F\bigl((1,2,3),3\bigr)+F\bigl((0,2,4),3\bigr)+F\bigl((1,2,4),2\bigr)\\
&\qquad + F\bigl((0,2,5),2\bigr)+F\bigl((1,2,5),1\bigr).
\end{aligned}$$

We invite the reader to check that the right-hand side is in $\Delta^{(2,1,1)}\operatorname{Sym}^5 E$; the final form is probably the most convenient for this.

(d) Taking $N=1$ we may identify $\operatorname{Sym}^0 E$ with $\mathbb{F}$ and $\Delta^{(2)}\operatorname{Sym}^d E$ with the symmetric tensors inside $\operatorname{Sym}^d E \otimes \operatorname{Sym}^d E$. The map $\varphi : \bigwedge^2 \operatorname{Sym}^{d+1} E \to$

$\Delta^{(2)} \operatorname{Sym}^d E$ is then defined by

$$\varphi(X^{d+1-k}Y^k \wedge X^{d+1-\ell}Y^\ell) = \sum_{k \le i < \ell} X^{d-i}Y^i \otimes X^{d-(k+\ell-1-i)}Y^{k+\ell-1-i}.$$

It is clear from the powers of Y in the tensor factors on the right-hand side then the right-hand side is a symmetric tensor, and so lies in $\Delta^{(2)} \operatorname{Sym}^d E$. This is an example of the Wronskian isomorphism mentioned at the start of the introduction, between the symmetric and exterior powers of symmetric powers of E.

2. The map φ is an $\mathrm{SL}_2(\mathbb{F})$-isomorphism

2.1. **The image of φ.** As defined φ has codomain $\bigwedge^N \operatorname{Sym}^d E \otimes \operatorname{Sym}^d E$.

Lemma 2.1. *The image of φ is contained in $\Delta^{(2,1^{N-1})} \operatorname{Sym}^d E$.*

Proof. Let $0 \le s \le N-1$ and let $\mathbf{k} \in I^{(d+1)}(N+1)$. Let Ω be the subset of

$$\mathcal{B}(\mathbf{k}) \times [0, d+1) = [k_1, k_2) \times \cdots \times [k_N, k_{N+1}) \times [0, d+1)$$

of all tuples $(i_1, \ldots, i_\alpha, j)$ such that $i_1 + \cdots + i_\alpha + j = s + |\mathbf{k}| - N$. Writing elements of Ω as $(\mathbf{i}, j)$, we have, using the notation of (1.3),

$$\mu_N \varphi\big(X^{N-1-s}Y^s \otimes F_\wedge^{(d+1)}(\mathbf{k})\big) = \sum_{(\mathbf{i},j)\in\Omega} \mu_N F(\mathbf{i}, j). \tag{2.1}$$

By the definition of $\Delta^{(2,1^{N-1})} \operatorname{Sym}^d E$ from (1.2), it suffices to show that the right-hand side vanishes. Our proof uses an involution on Ω closely related to the partition used to prove Lemma 1.3. First observe that if $(\mathbf{i}, j) \in \Omega$ then $j = s + |\mathbf{k}| - N - |\mathbf{i}|$ and since $i_\alpha \ge k_\alpha$ for each $0 \le \alpha \le N$, it follows that

$$j \le s + |\mathbf{k}| - N - k_1 - \cdots - k_N = s + k_{N+1} - N < k_{N+1}.$$

Similarly

$$j \ge s + |\mathbf{k}| - N - (k_2 + 1) - \cdots - (k_{N+1} + 1) = s + k_1 \ge k_1.$$

Thus, given $(\mathbf{i}, j) \in \Omega$, there exists a unique $0 \le \alpha \le N$ such that $k_\alpha \le j < k_{\alpha+1}$. We send $(\mathbf{i}, j)$ to $\big(i_1, \ldots, j, \ldots i_N, i_\alpha\big)$ where j appears in position α, so j replaces i_α. It is clear this defines an involution in which $(\mathbf{i}, j)$ is a fixed point if and only if $i_\alpha = j$. Since $(i_1, \ldots, j, \ldots, i_N, i_\alpha)$ and $(i_1, \ldots, i_\alpha, \ldots, i_N, j)$ are either equal or differ by a transposition, their contributions to the sum in (2.1) cancel. □

2.2. **φ is an $\mathrm{SL}_2(\mathbb{F})$-homomorphism.** For $\gamma \in \mathbb{N}$ we denote by $\mathbf{u}^{(\gamma)}$ the unit vector $(0, \ldots, 1, \ldots, 0)$ where the non-zero entry is in position γ; the length is always N or $N+1$ and will always be clear from context.

Reduction. We recall the technical trick in [McDW, §4.2] used to pass from $\mathrm{SL}_2(\mathbb{F})$ to $\mathrm{SL}_2(\mathbb{C})$. First notice that φ is a map of vector spaces, but it is defined over the integers. Let $\gamma \in \mathbb{F}$ be an arbitrary element and $U_\gamma = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$. The elements U_γ and their transposes generate $\mathrm{SL}_2(\mathbb{F})$. Checking that φ intertwines the action of U_γ (or its transpose) amounts to an equality of polynomials in γ with coefficients in the image of $\mathbb{Z}$ in $\mathbb{F}$. Clearly, it suffices to check that this equality holds over the polynomial ring $\mathbb{Z}[\gamma]$. For this, in turn, it suffices to prove the equality for *any* transcendental element γ in any field containing $\mathbb{Z}$ as a subring. Proving the result for $\mathrm{SL}_2(\mathbb{C})$ certainly implies the latter condition. A basic fact from Lie theory (see for instance [FH, Ch. 8]) then reduces the question to proving that φ commutes with the Lie algebra generators e and f of $\mathsf{sl}_2(\mathbb{C})$, defined on the X, Y basis of E by the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.2}$$

Their action on $\mathrm{Sym}^d E$ is given by $e \cdot v = X\frac{\mathrm{d}v}{\mathrm{d}Y}$ and $f \cdot v = Y\frac{\mathrm{d}v}{\mathrm{d}X}$. Their action on $\bigwedge^R \mathrm{Sym}^c E$ and on $\Delta^{(2,1^{N-1})}\,\mathrm{Sym}^d E$ is then given by the usual multilinear rule for Lie algebra actions, coming ultimately from

$$x \cdot (u \otimes v) = (x \cdot u) \otimes v + u \otimes (x \cdot v). \tag{2.3}$$

We state it below using the notation for the unit vectors $\mathbf{u}^{(\gamma)}$ just defined:

$$e \cdot F_\wedge^{(d+1)}(\mathbf{k}) = \sum_{\alpha=1}^{N+1} k_\alpha F_\wedge^{(d+1)}(\mathbf{k} + \mathbf{u}^{(\alpha)}) \tag{2.4}$$

$$f \cdot F_\wedge^{(d+1)}(\mathbf{k}) = \sum_{\alpha=1}^{N+1} (d+1-k_\alpha) F_\wedge^{(d+1)}(\mathbf{k} - \mathbf{u}^{(\alpha)})$$

$$e \cdot F^w(\mathbf{i}) = \sum_{\beta=1}^{N} i_\alpha F_\wedge^{w+1}(\mathbf{i} + \mathbf{u}^{(\beta)}) + \big(w - |\mathbf{i}|\big) F^{w+1}(\mathbf{i}) \tag{2.5}$$

$$f \cdot F^w(\mathbf{i}) = \sum_{\beta=1}^{N} (d-i_\alpha) F_\wedge^{w-1}(\mathbf{i} - \mathbf{u}^{(\beta)}) + \big(d - w + |\mathbf{i}|\big) F^{w-1}(\mathbf{i})$$

for $\mathbf{k} \in I_{N+1}^{(d+1)}$ and $\mathbf{i} \in I_N^{(d)}$. Here we use the convention that if $\mathbf{k} \pm \mathbf{u}^{(\alpha)} \notin I_R^{(c)}$ because $k_\alpha = 0$ or $k_\alpha > c$ then $F_\wedge^{(c)}(\mathbf{k} \pm \mathbf{u}^{(\alpha)}) = 0$, and similarly for $\mathbf{i} \pm \mathbf{u}^{(\beta)}$.

Technical lemma. The following lemma is a key step in the calculation that φ commutes with the Lie algebra action of e.

Lemma 2.2. *Let $0 \le s \le N-1$. Let $\mathbf{k} \in I^{(d+1)}(N+1)$ be strictly increasing. Then for any $v \ge |\mathbf{k}| - 1$ we have*

$$\sum_{\alpha=1}^{N+1} \sum_{\mathbf{i}\in\mathcal{B}(\mathbf{k}-\mathbf{u}^{(\alpha)})} k_\alpha F^v(\mathbf{i}) = \sum_{\beta=1}^{N} \sum_{\mathbf{j}\in\mathcal{B}(\mathbf{k})-\mathbf{u}^{(\beta)}} (j_\beta+1)F^v(\mathbf{j}) + \sum_{\mathbf{j}\in\mathcal{B}(\mathbf{k})} \big(|\mathbf{k}|-N-|\mathbf{j}|\big)F^v(\mathbf{j}).$$

Proof. Given $x \in \{0,\ldots,d\}$ and $1 \le \alpha \le N$, we set

$$\mathcal{C}^{(x)}_\alpha(\mathbf{k}) = [k_1,k_2) \times \cdots \times [k_{\alpha-1},k_\alpha) \times \{x\} \times [k_{\alpha+1},k_{\alpha+2}) \times \cdots \times [k_N,k_{N+1}).$$

where $\{x\}$ in position α replaces the interval $[k_\alpha,k_{\alpha+1})$ in position α of the product defining $\mathcal{B}(\mathbf{k})$ in (1.7). Observe that

$$\begin{aligned}\mathcal{B}(\mathbf{k}-\mathbf{u}^{(1)}) &= [k_1-1,k_2)\times[k_2,k_3)\times\cdots\times[k_N,k_{N+1}) = \mathcal{B}(\mathbf{k}) \cup \mathcal{C}^{(k_1-1)}_1,\\ \mathcal{B}(\mathbf{k}-\mathbf{u}^{(N+1)}) &= [k_1,k_2)\times\cdots\times[k_{N-1},k_N)\times[k_N,k_{N+1}-1) = \mathcal{B}(\mathbf{k}) \setminus \mathcal{C}^{(k_{N+1}-1)}_N\end{aligned}$$

and, if $2 \le \alpha \le N$, then

$$\begin{aligned}\mathcal{B}(\mathbf{k}-\mathbf{u}^{(\alpha)}) &= [k_1,k_2)\times\cdots\times[k_{\alpha-1},k_\alpha-1)\times[k_\alpha-1,k_\alpha)\times\cdots\times[k_N,k_{N+1})\\ &= \mathcal{B}(\mathbf{k}) \cup \mathcal{C}^{(k_\alpha-1)}_\alpha(\mathbf{k}) \setminus \mathcal{C}^{(k_\alpha-1)}_{\alpha-1}(\mathbf{k}). \qquad (2.6)\end{aligned}$$

Thus by setting $\mathcal{C}^{(x)}_0(\mathbf{k}) = \mathcal{C}^{(x)}_{N+1}(\mathbf{k}) = \varnothing$, we may unify the cases so that (2.6) holds for all $1 \le \alpha \le N+1$. By (2.6), the left-hand side in the lemma is

$$|\mathbf{k}| \sum_{\mathbf{i}\in\mathcal{B}(\mathbf{k})} F^v(\mathbf{i}) + \sum_{\alpha=1}^{N+1} \sum_{\mathbf{i}\in\mathcal{C}^{(k_\alpha-1)}_\alpha(\mathbf{k})} k_\alpha F^v(\mathbf{i}) - \sum_{\alpha=1}^{N+1} \sum_{\mathbf{i}\in\mathcal{C}^{(k_\alpha-1)}_{\alpha-1}(\mathbf{k})} k_\alpha F^v(\mathbf{i}). \qquad (2.7)$$

Similarly to (2.6) we have

$$\begin{aligned}\mathcal{B}(\mathbf{k}) - \mathbf{u}^{(\beta)} &= [k_1,k_2) \times \cdots \times [k_\beta-1,k_{\beta+1}-1) \times \cdots \times [k_N,k_{N+1})\\ &= \mathcal{B}(\mathbf{k}) \cup \mathcal{C}^{(k_\beta-1)}_\beta(\mathbf{k}) \setminus \mathcal{C}^{(k_{\beta+1}-1)}_\beta(\mathbf{k}). \qquad (2.8)\end{aligned}$$

By (2.8) the first summand in the right side in the lemma is

$$\sum_{\beta=1}^{N}\Big(\sum_{\mathbf{j}\in\mathcal{B}(\mathbf{k})} (j_\beta+1)F^v(\mathbf{j}) + \sum_{\mathbf{j}\in\mathcal{C}^{(k_\beta-1)}_\beta(\mathbf{k})} (k_\beta-1+1)F^v(\mathbf{j}) - \sum_{\mathbf{j}\in\mathcal{C}^{(k_{\beta+1}-1)}_\beta} (k_{\beta+1}-1+1)F^v(\mathbf{j})\Big).$$

Since $\sum_{\beta=1}^N (j_\beta+1) = |\mathbf{j}| + N$, and the second summand on the right-hand side is $\sum_{\mathbf{j}\in\mathcal{B}(\mathbf{k})}\big(|\mathbf{k}|-N-|\mathbf{j}|\big)F^v(\mathbf{j})$, the right-hand side in the lemma simplifies to

$$|\mathbf{k}| \sum_{\mathbf{j}\in\mathcal{B}(\mathbf{k})} F^v(\mathbf{j}) + \sum_{\beta=1}^{N} \sum_{\mathbf{j}\in\mathcal{C}^{(k_\beta-1)}_\beta} k_\beta F^v(\mathbf{j}) - \sum_{\beta=1}^{N} \sum_{\mathbf{j}\in\mathcal{C}^{(k_{\beta+1}-1)}_\beta} k_{\beta+1} F^v(\mathbf{j}). \qquad (2.9)$$

The lemma now follows by comparing (2.7) and (2.9); the three summands agree in the order written. $\square$

The map φ commutes with e. The Lie algebra element $e \in \mathsf{sl}_2(\mathbb{C})$ acts on $\mathrm{Sym}^d E$ by $e \cdot X^{d-j}Y^j = jX^{d-j+1}Y^{j-1}$.

Lemma 2.3. *The map φ defined over the complex numbers commutes with the Lie algebra action of $e \in \mathsf{sl}_2(\mathbb{C})$.*

Proof. We compare $e \cdot \varphi(x)$ and $\varphi(e \cdot x)$ for x in the canonical basis of $\mathrm{Sym}^{N-1}E \otimes \bigwedge^{N+1} \mathrm{Sym}^{d+1}E$. Let $0 \le s \le N-1$ and let $\mathbf{k} \in I^{(d+1)}_{N+1}$ be strictly increasing. For ease of notation we set $w = s + |\mathbf{k}| - N$. By (2.3) and (2.4) and the definition of φ in Definition 1.5, then the technical lemma to obtain the third equality, and finally (2.5) we have

$$\begin{aligned}
&\varphi\big(e \cdot (X^{N-1-s}Y^s \otimes F^{(d+1)}_\wedge(\mathbf{k})\big) \\
&\quad = \varphi\Big(sX^{N-s}Y^{s-1} \otimes F^{(d+1)}_\wedge(\mathbf{k}) + X^{N-1-s}Y^s \otimes \sum_{\alpha=1}^{N+1} k_\alpha F^{(d+1)}_\wedge(\mathbf{k} - \mathbf{u}^{(\alpha)})\Big) \\
&\quad = s \sum_{\mathbf{i}\in\mathcal{B}(\mathbf{k})} F^{s-1+|\mathbf{k}|-N}(\mathbf{i}) + \sum_{\alpha=1}^{N+1} k_\alpha \sum_{\mathbf{i}\in B(\mathbf{k}-\mathbf{u}^{(\alpha)})} F^{s+|\mathbf{k}|-1-N}(\mathbf{i}) \\
&\quad = \sum_{\beta=1}^{N} \sum_{\mathbf{j}\in B(\mathbf{k})-\mathbf{u}^{(\beta)}} (j_\beta+1)F^{w-1}(\mathbf{j}) + \sum_{\mathbf{j}\in B(\mathbf{k})} \big(s + |\mathbf{k}| - N - |\mathbf{j}|\big)F^{w-1}(\mathbf{j}) \\
&\quad = \sum_{\beta=1}^{N} \sum_{\mathbf{i}\in B(\mathbf{k})} i_\beta F^{w-1}(\mathbf{i} - \mathbf{u}^{(\beta)}) + \sum_{\mathbf{j}\in B(\mathbf{k})} \big(s + |\mathbf{k}| - N - |\mathbf{j}|\big)F^{w-1}(\mathbf{j}) \\
&\quad = e \cdot \sum_{\mathbf{j}\in B(\mathbf{k})} F^{s+|\mathbf{k}|-N}(\mathbf{j}) \\
&\quad = e \cdot \varphi\big(X^{N-1-s}Y^s \otimes F^{(d+1)}_\wedge(\mathbf{k})\big).
\end{aligned}$$

□

Duality. To show that φ commutes with f we use a duality argument. This appears to the authors to be more conceptual and involve less calculation than adapting the proof already given for e, although this would also be possible. Let $\mathbf{e} = (d+1, \ldots, d+1) \in I^{(d+1)}_{N+1}$ and define $\tau \in \mathrm{End}\big(\mathrm{Sym}^{N-1} E \otimes \bigwedge \mathrm{Sym}^{N+1} \mathrm{Sym}^{d+1}(E)\big)$ by linear extension of

$$\tau\big(X^{N-1-j}Y^j \otimes F^{(d+1)}_\wedge(\mathbf{i})\big) = X^jY^{N-1-j} \otimes F^{(d+1)}_\wedge(\mathbf{e} - \mathbf{i}).$$

Let $\mathbf{d} = (d, \ldots, d) \in I^{(d)}_N$ and define $\tau' \in \mathrm{End}\big(\bigwedge^N \mathrm{Sym}^d \otimes \mathrm{Sym}^d E\big)$ by linear extension of

$$\tau'\big(F^{(d)}_\wedge(\mathbf{j}) \otimes X^{d-\ell}Y^\ell\big) = F^{(d)}_\wedge(\mathbf{d} - \mathbf{j}) \otimes X^\ell Y^{d-\ell}.$$

Lemma 2.4. *We have $e\tau = \tau f$, $\tau' e = f\tau'$ and $\tau'\varphi = \pm\varphi\tau$.*

Proof. Observe that τ and τ' are defined by multilinear extension of the maps $\theta_c : \mathrm{Sym}^c E \to \mathrm{Sym}^c E$ defined on the canonical basis by

$$\theta_c(X^{c-j}Y^j) = X^jY^{c-j}.$$

Since $\theta_c\big(e \cdot X^{c-j}Y^j)\big) = \theta_c(jX^{c-j+1}Y^{j-1}) = jX^{j-1}Y^{c-j+1} = f \cdot X^jY^{c-j} = f \cdot \big(\theta_c(X^{c-j}Y^j)\big)$ we have $\theta_c\, e = f\theta_c$. By multilinearity, this implies the first two equations in the lemma. For the third, let ε_R denote the sign of the permutation of $\{1,\ldots,R\}$ reversing the positions in an R-tuple. (Thus $\varepsilon_R = -1$ if $R \equiv 2, 3 \bmod 4$, and otherwise $\varepsilon_R = 1$.) Let $0 \le s \le N-1$, let $\mathbf{k} \in I^{(d+1)}_{N+1}$ be strictly increasing, and let $\mathbf{k}^{\mathrm{rev}} = (k_{N+1},\ldots,k_1)$ be the reverse of $\mathbf{k}$. Set

$$w = N-1-s+(N+1)(d+1)-|\mathbf{k}|-N = dN+N+d-s. \qquad (2.10)$$

Observe that $\mathbf{e}-\mathbf{k}^{\mathrm{rev}}$ is strictly increasing, $|\mathbf{e}-\mathbf{k}^{\mathrm{rev}}| = (d+1)(N+1)-|\mathbf{k}|$ and, by (1.7),

$$\mathcal{B}(\mathbf{e}-\mathbf{k}^{\mathrm{rev}}) = [d+1-k_{N+1}, d+1-k_N) \times \cdots \times [d+1-k_2, d+1-k_1).$$

Thus $(i_1,\ldots,i_N) \in \mathcal{B}(\mathbf{e}-\mathbf{k}^{\mathrm{rev}})$ if and only if $(d+1-i_N,\ldots,d+1-i_1) \in (k_1,k_2] \times \cdots \times (k_N,k_{N+1}]$, so if and only if $(d-i_n,\ldots,d-i_1) \in \mathcal{B}(\mathbf{k}) = [k_1,k_2) \times \cdots \times [k_N,k_{N+1})$. Using this to step from line 3 to line 4 below, and the alternative definition of φ in (1.8) for the immediately preceding step, we have

$$\begin{aligned}
&\varphi\tau\big(X^{N-1-s}Y^s \otimes F^{(d+1)}_{\wedge}(\mathbf{k})\big)\\
&\quad= \varphi\big(X^sY^{N-1-s} \otimes \varepsilon_{N+1}F^{(d+1)}_{\wedge}(\mathbf{e}-\mathbf{k}^{\mathrm{rev}})\big)\\
&\quad= \sum_{\mathbf{i}\in\mathcal{B}(\mathbf{e}-\mathbf{k}^{\mathrm{rev}})} \varepsilon_{N+1}F(\mathbf{i}, w-|\mathbf{i}|)\\
&\quad= \varepsilon_{N+1}\sum_{\mathbf{j}\in\mathcal{B}(\mathbf{k})} F\big((d-j_N,\ldots,d-j_1), w-(dN-|\mathbf{j}|)\big)\\
&\quad= \varepsilon_N\varepsilon_{N+1}\tau'\Big(\sum_{\mathbf{j}\in\mathcal{B}(\mathbf{k})} F\big((j_1,\ldots,j_N), d-(w-(dN-|\mathbf{j}|))\big)\Big)\\
&\quad= \varepsilon_N\varepsilon_{N+1}\tau'\Big(\sum_{\mathbf{j}\in\mathcal{B}(\mathbf{k})} F\big((j_1,\ldots,j_N), s+N-|\mathbf{j}|\big)\\
&\quad= \varepsilon_N\varepsilon_{N+1}\tau'\varphi\big(X^{N-1-s}Y^s \otimes F^{(d+1)}_{\wedge}(\mathbf{k})\big)
\end{aligned}$$

where the penultimate equality uses (2.10). Since $\varepsilon_N\varepsilon_{N+1} \in \{-1,1\}$ only depends on N, this completes the proof. □

Proposition 2.5. *The map φ defined over the complex numbers is an $\mathsf{sl}_2(\mathbb{C})$-homomorphism.*

Proof. By Lemma 2.3, φ commutes with the Lie algebra action of e. By this lemma and Lemma 2.4 we have

$$\varphi f = \varphi\tau\tau f = \varphi\tau e\tau = \pm\tau'\varphi e\tau = \pm\tau' e\varphi\tau = \pm f\tau'\varphi\tau = f\varphi\tau\tau = f\varphi,$$

and so φ also commutes with the Lie algebra action of f. Since $\mathsf{sl}_2(\mathbb{C})$ is generated by e and f the proposition follows. □

2.3. **The map φ is an $\mathrm{SL}_2(\mathbb{F})$-isomorphism.** Fix $N \in \mathbb{N}$ and $d \in \mathbb{N}_0$. The canonical basis of $\mathrm{Sym}^{N-1}E \otimes \bigwedge^{N+1} \mathrm{Sym}^{d+1}E$ is indexed by pairs $(s, \mathbf{k})$ with $0 \le s \le N-1$ and $\mathbf{k} \in I^{(d+1)}_{N+1}$ strictly increasing. Whenever we write a pair $(s, \mathbf{k})$, it satisfies these conditions. By (1.8), the vectors

$$v_{(s,\mathbf{k})} = \sum_{\mathbf{i} \in \mathcal{B}(\mathbf{k})} F(\mathbf{i}, w - |\mathbf{i}|) \tag{2.11}$$

where $w = s + |\mathbf{k}| - N$ are the images under φ of the canonical basis of $\mathrm{Sym}^{N-1}E \otimes \bigwedge^{N+1} \mathrm{Sym}^{d+1}E$. Since we know by Lemma 2.1 that φ has image contained in $\Delta^{(2,1^{N-1})} \mathrm{Sym}^d E$ and by Lemma 1.4, this space has the same dimension as the domain of φ, to complete the proof that φ is an isomorphism of $\mathrm{SL}_2(\mathbb{F})$-representations, it suffices to show that the vectors $v_{(s,\mathbf{k})}$ are linearly independent.

The strategy of the rest of our proof involves two steps. The first one is to define leading terms that let us separate the sum (2.11) into sums where the middle indices $k_2, \ldots, k_N$ are fixed; for the second step, we assume these are fixed, and define additional leading terms in order to show linear independence. This is illustrated by the following example.

Example 2.6. Take $N = 2$ and any $d \in \mathbb{N}_0$. For $0 \le k_1 \le k_2 \le k_3 \le d+1$ we have

$$\begin{aligned} v_{(s,(k_1,k_2,k_3))} &= \sum_{\mathbf{i} \in [k_1,k_2) \times [k_2,k_3)} F\big((i_1, i_2), w - i_1 - i_2\big) \\ &= \sum_{\mathbf{i} \in [k_1,k_2) \times [k_2,k_3)} X^{d-i_1}Y^{i_1} \wedge X^{d-i_2}Y^{i_2} \otimes X^{d-(w-i_1-i_2)}Y^{w-i_1-i_2} \end{aligned}$$

where $w = s + k_1 + k_2 + k_3 - 2$. By (1.7), $\mathcal{B}(k_1, k_2, k_3) = [k_1, k_2) \times [k_2, k_3) \subseteq \{0, 1, \ldots, d\} \times \{0, 1, \ldots, d\}$. The diagram in Figure 1 shows three such sets visualized as boxes in the plane. Observe that each box $\mathcal{B}(k_1, k_2, k_3)$ meets the super-diagonal $\{(x, x+1) : x \in \mathbb{R}^{\ge 0}\}$ in a unique point, namely $(k_2 - 1, k_2)$, and this intersection corresponds to a summand

$$F\big((k_2 - 1, k_2), s + k_1 + k_3 - k_2 - 1\big)$$

in $v_{(s,(k-1,k_2,k_3))}$. (These intersections are shown by white dots in the diagram.) Therefore the summands in each $v_{(s,\mathbf{k})}$ determine k_2 and any linear independency between these vectors may be assumed to hold *for fixed* k_2.

Now consider the two boxes in the diagram with $k_2 = 5$. The top-left corner of each box is the unique point maximizing the difference of the two coordinates. Thus using (1.3) we have

$$\begin{aligned} v_{(s,(2,5,8))} &= F^{(d)}_{\wedge}(2,7) \otimes X^{d-(s+4)}Y^{s+4} + \cdots \\ v_{(s,(0,5,7))} &= F^{(d)}_{\wedge}(0,6) \otimes X^{d-(s+4)}Y^{s+4} + \cdots \end{aligned}$$

and generally $F_{\wedge}^{(d)}(2,7)$ appears as the tensor factor in a $v_{(s,(k_1,5,k_3))}$ with maximum coordinate difference if and only if $\mathbf{k} = (2,5,8)$. We may therefore use these top-left corners to define leading terms in the $v_{(s,\mathbf{k})}$ for fixed k_2 which uniquely determine $\mathbf{k}$ and demonstrate the linear independence of these vectors.

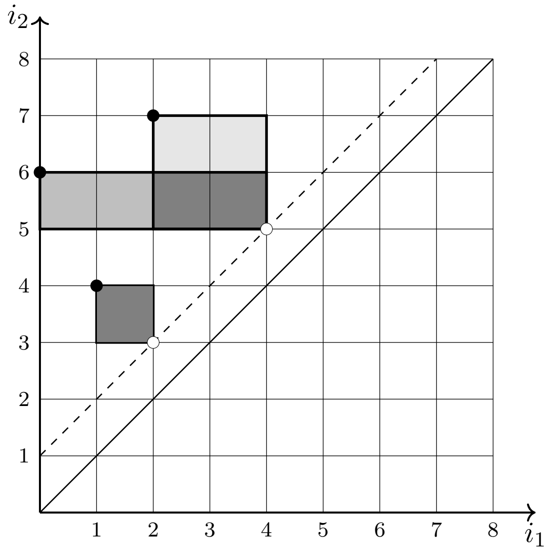

FIGURE 1. The three boxes $\mathcal{B}(2,5,8) = \{2,3,4\} \times \{5,6,7\}$, $\mathcal{B}(0,5,7) = \{0,1,2,3,4\} \times \{5,6\}$ and $\mathcal{B}(1,3,5) = \{1,2\} \times \{3,4\}$. The first two boxes intersect in the darker shader region. The dashed line is the super-diagonal $\{(x, x+1) : x \in \mathbb{R}^{\geq 0}\}$.

We begin the formal proof with the general version of the reduction in the first part of the example.

Lemma 2.7. *Let $\alpha_{(s,\mathbf{k})} \in \mathbb{F}$ for each pair $(s,\mathbf{k})$. Fix $(c_2, \ldots, c_M) \in I_{M-1}^{(d+1)}$ where $M \leq N$. If $\sum_{(s,\mathbf{k})} \alpha_{(s,\mathbf{k})} v_{(s,\mathbf{k})} = 0$ then*

$$\sum_{\substack{(s,\mathbf{k}) \\ (k_2,\ldots,k_M)=(c_2,\ldots,c_M)}} \alpha_{(s,\mathbf{k})} v_{(s,\mathbf{k})} = 0$$

Proof. We work by induction on M. If $M = 1$ there is nothing to prove. When $M \geq 2$ we may assume by induction that

$$\sum_{\substack{(s,\mathbf{k})\\(k_2,\ldots,k_{M-1})=(c_2,\ldots,c_{M-1})}} \alpha_{(s,\mathbf{k})} v_{(s,\mathbf{k})} = 0 \tag{2.12}$$

Consider the subspace W of $\bigwedge^N \operatorname{Sym}^d E$ spanned by all $F_\wedge^{(d)}(\mathbf{i})$ for all strictly increasing $\mathbf{i}$ of the form

$$(i_1, \ldots, i_{M-1}, c_M - 1, c_M, i_{M+2}, \ldots, i_N)$$

such that $i_{M-1} < c_M - 1$ and $c_M < i_{M+2}$. We define a projection map $\pi : \bigwedge^N \operatorname{Sym}^d E \otimes \operatorname{Sym}^d E \to W \otimes \operatorname{Sym}^d E$ by

$$\pi(F_\wedge^{(d)}(\mathbf{i}) \otimes X^{d-j}Y^j) = \begin{cases} F_\wedge^{(d)}(\mathbf{i}) \otimes X^{d-j}Y^j & \text{if } F_\wedge^{(d)}(\mathbf{i}) \in W, \\ 0 & \text{otherwise.} \end{cases}$$

By definition, W contains $F_\wedge^{(d)}(\mathbf{i})$ if and only if $i_M = c_M - 1$ and $i_{M+1} = c_M$. Moreover, since in the product $B(\mathbf{k})$ the $M-1^{\text{th}}$ factor is $[k_{M-1}, k_M)$ and the M^{th} factor is $[k_M, k_{M+1})$, there *is* such an $\mathbf{i}$ with $\mathbf{i} \in B(\mathbf{k})$ if and only if $k_M = c_M$. Therefore the projection π of the left-hand side of (2.12) into $W \otimes \operatorname{Sym}^d E$ is

$$\sum_{\substack{(s,\mathbf{k})\\(k_2,\ldots,k_{M-1})=(c_2,\ldots,c_{M-1})\\k_M=c_M}} \alpha_{(s,\mathbf{k})} v_{(s,\mathbf{k})}.$$

By (2.12), this is the projection of the zero vector, and so is zero. This completes the inductive step. □

After this reduction we need only the following proposition.

Proposition 2.8. *For each fixed* $(c_2, \ldots, c_N)$ *the vectors* $v_{(s,\mathbf{k})}$ *with* $\mathbf{k} = (k_1, c_2, \ldots, c_N, k_{N+1})$ *are linearly independent.*

Proof. Suppose, for a contradiction, that

$$\sum_{\substack{0\leq s\leq N-1\\0\leq k_1<c_2\\c_N<k_{N+1}\leq d+1}} \alpha_{(s,k_1,k_{N+1})} v_{(s,(k_1,c_2,\ldots,c_N,k_{N+1}))} = 0 \tag{2.13}$$

where the coefficients $\alpha_{(s,k_1,k_{N+1})} \in \mathbb{F}$ are not all zero. Choose k_1' minimal such that $\alpha_{(s,k_1',k_{N+1})}$ is non-zero for some s and k_{N+1}, and then choose k_{N+1}' maximal such that $\alpha_{(s,k_1',k_{N+1}')}$ is non-zero for some s. Finally choose any s' such that $\alpha_{(s',k_1',k_{N+1})}$ is non-zero. Observe that the canonical basis vector

$$F_\wedge^{(d)}(k_1', c_2, \ldots, c_{N-1}, k_{N+1}' - 1) \otimes X^{d-(w-s')}Y^{w-s'}, \tag{2.14}$$

where $w = s' + c_N + 1$, is a summand of $v_{(s',\mathbf{k})}$. By minimality of k_1 and maximality of $k_{N+1} - 1$ if either $k_1 \neq k_1'$ or $k_{N+1} \neq k_{N+1}'$ then it is not a summand of $v_{(s,(k_1',c_2,\ldots,c_N,k_{N+1}'))}$ for any s. Therefore the coefficient of the canonical basis element of $\bigwedge^N \mathrm{Sym}^d E \otimes \mathrm{Sym}^d$ in (2.14) in the left-hand side of (2.13) is non-zero. This contradicts the right-hand side of (2.13). $\square$

Given the summary at the start of this subsection, the next proposition completes the proof that φ is an isomorphism of representations of $\mathrm{SL}_2(\mathbb{F})$.

Proposition 2.9. *The vectors $v_{(s,\mathbf{k})}$ for $0 \le s \le N-1$ and $\mathbf{k} \in I^{(d+1)}_{N+1}$ are linearly independent.*

Proof. Suppose that $\sum_{(s,\mathbf{k})} \alpha_{(s,\mathbf{k})} v_{(s,\mathbf{k})} = 0$ where not all the coefficients $\alpha_{(s,\mathbf{k})}$ are zero. Choose $(c_2, \ldots, c_N) \in I^{(d+1)}_{M-1}$ such that $\alpha_{(s,(k_1,c_2,\ldots,c_N,k_{N+1}))} \neq 0$ for some k_1, k_{N+1} and s. By Lemma 2.7 there is a non-trivial linear dependency involving only those $v_{(s,\mathbf{k})}$ such that $\mathbf{k}$ is of the special form $(k_1, c_2, \ldots, c_N, k_{N+1})$. But this contradicts Proposition 2.8. $\square$

3. Final remarks

In this section we first show that the $\mathrm{SL}_2(\mathbb{F})$ isomorphism in Theorem 1.1 becomes a $\mathrm{GL}_2(\mathbb{F})$-isomorphism provided a suitable power of the determinant is introduced. (This is typical of the general theory: see [dBPW21, §3.3].) We then obtain identity (1.1) by taking characters. We finish with a conjectured generalization of Theorem 1.1.

We denote the 1-dimensional determinant representation of $\mathrm{GL}_2(\mathbb{F})$ by det.

Corollary 3.1. *Let $N \in \mathbb{N}$ and let $d \in \mathbb{N}_0$. The map φ defined in Definition 1.5 is an isomorphism of $\mathrm{GL}_2(\mathbb{F})$-representations*

$$\mathrm{Sym}^{N-1} E \otimes \bigwedge^{N+1} \mathrm{Sym}^{d+1} E \cong \det^N \otimes \Delta^{(2,1^{N-1})} \mathrm{Sym}^d E.$$

Proof. Let K be the algebraic closure of $\mathbb{F}$. Let $\widetilde{E} = E \otimes_{\mathbb{F}} K$. It is sufficient to prove that the map

$$\widetilde{\varphi} : \mathrm{Sym}^{N-1} \widetilde{E} \otimes \mathrm{Sym}^{N+1} \mathrm{Sym}^{d+1} \widetilde{E} \to \det^N \otimes \Delta^{(2,1^{N-1})} \mathrm{Sym}^d \widetilde{E}$$

is a $\mathrm{GL}_2(K)$-isomorphism, since φ is defined with coefficients in the prime subfield of $\mathbb{F}$, and so via the inclusion $E \mapsto E \otimes 1 \subseteq E \otimes_{\mathbb{F}} K$, the map $\widetilde{\varphi}$ restricts to φ. By Theorem 1.1 for the field K, the map $\widetilde{\varphi}$ is an $\mathrm{SL}_2(K)$-homomorphism. Now, because K is algebraically closed, and so every element of K has a square root in K, we have

$$\mathrm{GL}_2(K) = \left\langle \mathrm{SL}_2(K), \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} : \alpha \in K \backslash \{0\} \right\rangle.$$

It therefore suffices to prove that $\widetilde{\varphi}$ commutes with the action of the diagonal matrices αI for $\alpha \in K$. Using the canonical bases of the domain and codomain of $\widetilde{\varphi}$, one sees that on the domain αI acts as $\alpha^{N-1+(N+1)(d+1)} = \alpha^{(N+1)d+2N}$ and on the codomain αI acts as $\alpha^{(N+1)d} \det(\alpha I)^N = \alpha^{(N+1)d} \alpha^{2N}$. Since the exponents agree, this completes the proof. $\square$

We now prove identity (1.1). Recall that s_λ is the Schur function canonically labelled by the partition λ. It is immediate from the combinatorial definition of Schur functions (see for instance [S99, Definition 7.10.1]) that $s_\lambda(1, q, \ldots, q^d)$ is the generating function enumerating semistandard tableaux of shape λ with entries from $\{0, 1, \ldots, d\}$ by their sum of entries. This gives a combinatorial interpretation of the right-hand side in (1.1) and in Corollary 3.2 below. Perhaps the most natural interpretation of the q-binomial coefficient $\left[{a \atop b}\right]_q$ is that $q^{\frac{b(b-1)}{2}}\left[{a \atop b}\right]_q$ is the generating function enumerating b-subsets of $\{0, \ldots, a-1\}$ by their sum of entries. Thus, by identifying semistandard tableaux of shape (1^{N+1}) with the subset of their entries, we deduce that

$$q^{\frac{(N+1)N}{2}} \begin{bmatrix} d+2 \\ N+1 \end{bmatrix}_q = s_{(1^{N+1})}(1, q, \ldots, q^{d+1}). \tag{3.1}$$

For further background on q-binomial coefficients, including the theorem that $\left[{a \atop b}\right]_q$ is the generating function enumerating partitions in the $b \times (a-b)$ box by their size, we refer the reader to [S11, §1.7].

Corollary 3.2. *For any $N \in \mathbb{N}$ and $d \in \mathbb{N}_0$ we have*

$$q^{\frac{N(N-1)}{2}} [N]_q \begin{bmatrix} d+2 \\ N+1 \end{bmatrix}_q = s_{(2,1^{N-1})}(1, q, \ldots, q^d)$$

Proof. It suffices to prove the identity when q is a non-zero complex number. It is clear from the canonical basis $X^{n-1}, X^{n-1}Y, \ldots, Y^{N-1}$ of $\mathrm{Sym}^{N-1}E$ that $[N]_q = 1 + q + \cdots + q^{N-1}$ is the character of $\mathrm{Sym}^{N-1}E$ evaluated at the diagonal matrix D in $\mathrm{GL}_2(\mathbb{C})$ with entries 1 and q. By [PW21, (2.8)], $s_{(1^{N+1})}(1, q, \ldots, q^{d+1})$ and $s_{(2,1^{N-1})}(1, q, \ldots, q^d)$ are the characters of the $\mathrm{GL}_2(\mathbb{C})$-representations $\bigwedge^{N+1} \mathrm{Sym}^{d+1}E$ and $\Delta^{(2,1^{N-1})} \mathrm{Sym}^d E$, also evaluated at D. Therefore, by (3.1), the character of $\mathrm{Sym}^{N-1}E \otimes \bigwedge^{N+1} \mathrm{Sym}^{d+1}E$ evaluated at D is $[N]_q\, q^{(N+1)N/2} \left[{d+2 \atop N+1}\right]_q$. By Corollary 3.1 this representation is isomorphic to $\det^N \otimes \Delta^{(2,1^{N-1})}E$. Hence equating the character values we obtain

$$[N]_q\, q^{\frac{(N+1)N}{2}} \begin{bmatrix} d+2 \\ N+1 \end{bmatrix}_q = q^N s_{(2,1^{N-1})}(1, q, \ldots, q^d).$$

The result follows by cancelling q^N from each side. $\square$

Our main result, Theorem 1.1, is the special case when $M = 2$ of the following conjecture.

Conjecture 3.3. *Let M, $N \in \mathbb{N}$. There is an isomorphism of* $\mathrm{SL}_2(\mathbb{F})$-*representations*

$$\bigwedge^{M-1} \mathrm{Sym}^{M+N-3}E \otimes \bigwedge^{M+N-1} \mathrm{Sym}^{M+d-1}E \cong \Delta^{(M,1^{N-1})} \mathrm{Sym}^d E.$$

If $M = 1$, then the first factor is $\mathbb{F}$ and since $\Delta^{(1^N)}V = \bigwedge^N V$, both sides in the claimed isomorphism are $\bigwedge^N \mathrm{Sym}^d E$. If $N = 1$ then since $\mathrm{Sym}^{M-2}E$ is $(M-1)$-dimensional, and so $\bigwedge^{M-1} \mathrm{Sym}^{M-2}E$ is the determinant representation of $\mathrm{SL}_2(\mathbb{F})$, which is trivial, and $\Delta^{(M)}V = \mathrm{Sym}_M V$, the claimed isomorphism is $\bigwedge^M \mathrm{Sym}^{M+d-1}E \cong \mathrm{Sym}_M \mathrm{Sym}^d E$. An explicit isomorphism from the right-hand side to the left-hand side is given by Theorem 1.4 in [McDW]. More broadly, it would be interesting to have field-independent results on the endomorphism rings of the two sides in Conjecture 3.3.

Acknowledgements

The first author would like to thank his advisor Mikhail Khovanov for valuable discussions. The second author gratefully acknowledges financial support from the Heilbronn Institute for Mathematical Research, Bristol, UK.

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