NOTES ON POLYNOMIAL REPRESENTATIONS OF GENERAL LINEAR GROUPS

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Contents

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Notation: A composition of an non-negative integer n is a tuple of non-negative integers whose sum is n.

1. Definition of polynomial representations

Let F be an infinite field. For each pair (i,j) with $1 \le i,j \le n$, let $X_{ij} : \mathrm{GL}_n(F) \to F$ be the coordinate function sending a matrix x to its entry x_{ij}. Let $F[X_{ij}]$ denote the algebra generated by these functions. As this notation suggests, we identify $F[X_{ij}]$ with the polynomial ring in n^2 indeterminants; this is admissible because the field F is infinite.

Let V be a finite-dimensional F-representation of $\mathrm{GL}_n(F)$. We say that V is a *polynomial representation* if there is a basis $v_1, \ldots, v_d$ of V such that the functions $f_{ab} : \mathrm{GL}_n(F) \to F$ for $1 \le a, b \le d$ defined by

$$gv_b = \sum_a f_{ab}(g) v_a \quad \text{for } g \in \mathrm{GL}_n(F) \tag{1}$$

lie in the polynomial algebra $F[X_{ij}]$. Thus a representation of $\mathrm{GL}_n(F)$ is polynomial if and only if the action of each $g \in \mathrm{GL}_n(F)$ on V is given by a fixed family of polynomials in the entries of g.

1.1. **Example.** Let $n = 2$ and let E be a 2-dimensional F-vector space with basis e_1, e_2. The symmetric square $\mathrm{Sym}^2 E$ is then a representation of $\mathrm{GL}_2(F)$. With respect to the basis e_1^2, e_1e_2, e_2^2 of $\mathrm{Sym}^2 E$, the matrix

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \in \mathrm{GL}_2(F)$$

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acts on $\mathrm{Sym}^2 E$ as

$$\begin{pmatrix} \alpha^2 & \alpha\gamma & \gamma^2 \\ 2\alpha\beta & \alpha\delta + \beta\gamma & 2\gamma\delta \\ \beta^2 & \beta\delta & \delta^2 \end{pmatrix}.$$

The representation $\mathrm{Sym}^2 E$ is therefore polynomial, with $f_{11} = X_{11}^2$, $f_{12} = X_{11}X_{12}$, $f_{13} = X_{12}^2$, and so on.

1.2. **Non-example.** Let $n = 2$ and let $\rho : \mathrm{GL}_2(F) \to \mathrm{GL}_1(F) = F^\times$ be the representation defined by $\rho(g) = (\det g)^{-1}$. If ρ is a polynomial representation, then the function $\mathrm{GL}_2(F) \to F$ defined by

$$g \mapsto (\det g)^{-1} = (X_{11}(g)X_{22}(g) - X_{12}(g)X_{21}(g))^{-1}$$

would be a polynomial in the X_{ij}. We leave it to the reader to check that this is impossible.

For a more exciting non-example, keep $n = 2$ and suppose that $F = \mathbf{R}$. Define $\rho : \mathrm{GL}_2(\mathbf{R}) \to \mathrm{GL}_2(\mathbf{R})$ by

$$\rho \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} 1 & \log|\alpha\delta - \beta\gamma| \\ 0 & 1 \end{pmatrix}.$$

It is routine to check that ρ is a representation of $\mathrm{GL}_2(\mathbf{R})$. The function $f_{12} : \mathrm{GL}_2(\mathbf{R}) \to \mathbf{R}$ giving the entry in position $(1, 2)$ of a representing matrix is

$$g \mapsto \log|X_{11}(g)X_{22}(g) - X_{12}(g)X_{21}(g)|,$$

which is clearly some way from being a polynomial in the X_{ij}. By Remark (i) below, ρ is not a polynomial representation.

1.3. **Remarks.**

(i) If the functions f_{ab} defined by equation (1) are polynomials in the X_{ij} for one choice of basis of V, then they are polynomials for all choices of basis.

(ii) Suppose that $\rho : \mathrm{GL}_n(F) \to \mathrm{GL}_d(F)$ and $\sigma : \mathrm{GL}_d(F) \to \mathrm{GL}_e(F)$ are polynomial representations. Then the composition $\sigma \circ \rho : \mathrm{GL}_n(F) \to \mathrm{GL}_e(F)$ is also polynomial. Such compositions are known as *plethysms*.

(iii) In order to give some idea why we assume F to be infinite, it is useful to examine a result for which this assumption is essential.

Proposition 1. *Let E be an n-dimensional vector space and let $r \neq s$. The polynomial representations $E^{\otimes r}$ and $E^{\otimes s}$ of $\mathrm{GL}(E)$ have no common composition factors.*

Proof. Given a polynomial representation V of $\mathrm{GL}(E)$, let $\mathrm{cf}\, V \subseteq F[X_{ij}]$ be the linear span of the coefficient functions f_{ab} defined by (1). We note that if U is a subquotient of V then $\mathrm{cf}\, U \subseteq \mathrm{cf}\, V$. Hence, if $E^{\otimes r}$ and $E^{\otimes s}$ have a common composition factor, $\mathrm{cf}(E^{\otimes r}) \cap \mathrm{cf}(E^{\otimes s}) \neq 0$. But

$$\mathrm{cf}(E^{\otimes r}) = \{f \in F[X_{ij}] : \deg f = r\}.$$

so this is impossible. $\square$

The proof of the proposition clearly used our identification of $F[X_{ij}]$ with a polynomial ring; this fails when F is a finite field. Over $\mathbf{F}_q$ for instance, we have

$$X_{ij}^q = X_{ij} \quad \text{for all } i \text{ and } j.$$

And indeed when F is finite, the proposition is usually false. For example, if $V = \mathbf{F}_p$ is the natural representation of $\mathbf{F}_p^\times = \mathrm{GL}(\mathbf{F}_p)$, then $V^{\otimes p}$ is in fact *isomorphic* to V. Similarly, if $V = \mathbf{F}_3^2$ then one has

$$\phi_V^3 = \phi_V + 3\bar{\phi}_V$$

where ϕ_V denotes the Brauer character of V, so $V^{\otimes 3}$ and V have a common composition factor, namely V itself. (Note that in either case V is certainly polynomial. In fact, when F is finite *every* representation of $\mathrm{GL}_n(F)$ is polynomial, for the mundane reason that *every* function $F \to F$ is a polynomial.)

(iv) A more general context for polynomial representations is given by algebraic groups. Some remarks on this rather tricky subject are made in the Appendix. Green's lecture notes [2] provide a good introduction.

2. Weight spaces

Let T_n be the subgroup of $\mathrm{GL}_n(F)$ consisting of all its diagonal matrices. We shall write $t_1, \ldots, t_n$ for the entries on the diagonal of $t \in T_n$. Let V be a polynomial representation of $\mathrm{GL}_n(F)$. Given α a composition of r with at most n parts, let V_α be the subspace of V defined by

$$V_\alpha = \{v \in V : tv = t_1^{\alpha_1} \ldots t_n^{\alpha_n} v \text{ for all } t \in T_n\}.$$

We say that V_α is the *α-weight space* of V. Note that if we define polynomial representations of T_n by the obvious analogue of equation (1), then V_α is a polynomial representation of T_n.

It is an important fact that any polynomial representation of $\mathrm{GL}_n(F)$ decomposes as a direct sum of its weight spaces. This follows from the next theorem.

Theorem 2. *If V is a polynomial representation of T_n then V decomposes as a direct sum of weight spaces. That is,*

$$V = \bigoplus_\alpha V_\alpha$$

where the sum is over all compositions α with at most n parts.

An equivalent formulation is that (a) every simple polynomial representation of T_n is of the form $t \mapsto t_1^{\alpha_1} \ldots t_n^{\alpha_n}$ for some powers α_i, and (b) every representation of T_n is semisimple.

We first show that it suffices to prove the theorem in the case where $n = 1$. This gives the base case for an inductive argument. Suppose that the theorem is known for representations of T_{n-1}. Since T_n is abelian, each weight space for T_{n-1} is invariant under the action of the matrices

$$t(\alpha) = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \alpha \end{pmatrix}.$$

By the $n = 1$ case, we may decompose each T_{n-1}-weight space into common eigenspaces for the matrices $t(\alpha)$, such that in each eigenspace, $t(\alpha)$ acts as α^r for some $r \in \mathbf{N_0}$. This gives the inductive step.

There does not appear any really easy proof of the base case that works for a general infinite field F. (Although part (a) of the equivalent formulation given above follows quite easily from Schur's Lemma, part (b) seems harder.) We outline three possible approaches.

2.1. **Proof by Lie theory when $F = \mathbf{R}$.** Consider the function $s : \mathbf{R} \to \mathrm{GL}(V)$ defined by $s(x) = \rho(\mathrm{e}^x)$. Since

$$s(x+t) = \rho(\mathrm{e}^{x+t}) = \rho(\mathrm{e}^x)\rho(\mathrm{e}^t) = s(x)\rho(\mathrm{e}^t),$$

the derivative of s is given by

$$s'(x) = \rho'(1)s(x).$$

Let $T = \rho'(1) \in \mathrm{End}(V)$. The unique solution to the differential equation $s'(x) = Ts(x)$ is $s(x) = \exp(Tx)$. Hence

$$\rho(\mathrm{e}^x) = \exp(Tx).$$

Suppose that $\lambda \in \mathbf{C}$ is an eigenvalue of T. By passing to $V \otimes_{\mathbf{R}} \mathbf{C}$, and taking a suitable basis of the complexified space, we see that there is a coordinate function f of the form

$$f(\mathrm{e}^x) = \mathrm{e}^{\lambda x}.$$

By our assumption that ρ is polynomial, $\lambda \in \mathbf{N}_0$. Hence all the eigenvalues of T are real, and there is a basis of V in which ρ is a direct sum of representations of the form

$$\rho_\alpha(y) = \begin{pmatrix} y^\alpha & \star & \star & \dots & \star \\ & y^\alpha & \star & \dots & \star \\ & & \ddots & \vdots & \vdots \\ & & & y^\alpha & \star \\ & & & & y^\alpha \end{pmatrix}$$

for some $\alpha \in \mathbf{N}_0$.

It only remains to show that the above diagonal entries must vanish for all y. This may be deduced from the functional equation $\rho_\alpha(y^2) = \rho_\alpha(y)^2$. For example, if $f(y)$ is the coordinate function for position $(1, 2)$ we get

$$y^{2\alpha} f(y^2) = 2y^\alpha f(y) \quad \text{for all } y \in \mathbf{R},$$

from which it follows by comparing degrees that $f = 0$. (But note that if $\alpha = 0$ and f is not assumed to be polynomial, then $\log |x|$ is a possibility.)

2.2. **Ad-hoc argument.** We first assume that F is algebraically closed. It follows that this assumption F that has infinitely many elements of finite order: in fact provided the field characteristic does not divide m, the polynomial $X^m - 1$ has $\phi(m)$ roots in F of order m.

Let $x \in F^\times$ have finite order m. The minimal polynomial of $\rho(x) \in \mathrm{GL}(V)$ must divide $X^m - 1$ since

$$\rho(x)^m = \rho(x^m) = \rho(1) = 1.$$

Hence $X^m - 1$ splits as a product of linear factors in F. It follows that $\rho(x)$ acts diagonalisably on V.

An infinite family of commuting diagonalisable matrices acting on a finite-dimensional vector space may be simultaneously diagonalised. (This can be proved in much the same way as in the finite case by induction on the dimension of the space.) There is therefore a basis for V and polynomial functions $f_i : F^\times \to F$ such that

$$\rho(x) = \begin{pmatrix} f_1(x) & & \\ & \ddots & \\ & & f_d(x) \end{pmatrix}$$

for all x of finite order. It now follows easily that the f_i are of the form $f_i(x) = x^{\alpha_i}$ for some $\alpha_i \in \mathbf{N}_0$. The off-diagonal coordinate functions vanish on the infinite subset of F consisting of elements of finite order, so are identically zero.

We now prove the result when F is not necessarily algebraically closed. Let F^{alg} denote the algebraic closure of F. Any polynomial representation $\rho : \mathrm{GL}_n(F) \to \mathrm{GL}(V)$ can be extended to a map $\tilde{\rho} : \mathrm{GL}_n(F^{\mathrm{alg}}) \to \mathrm{GL}(V \otimes_F F^{\mathrm{alg}})$ since the matrix coefficients are given by polynomials. We have $\tilde{\rho}(gh) = \tilde{\rho}(g)\tilde{\rho}(h)$ for all $g, h \in \mathrm{GL}_n(F)$; since $\mathrm{GL}_n(F)$ is a Zariski dense subset of $\mathrm{GL}_n(F^{\mathrm{alg}})$, it follows that $\tilde{\rho}$ is a polynomial representation of $\mathrm{GL}_n(F)$. I am grateful to Darij Grinberg for suggesting this argument.

2.3. **Proof by comodules.** Using the language of comodules it is possible to give a remarkably quick proof of Theorem 2. What follows is based on Green's paper [1]. (See the appendix for undefined terms.)

Proposition 3. *Let R be a coalgebra which has a coalgebra decomposition*

$$R = \bigoplus_{n \in \mathbf{Z}} R_n.$$

If V is a comodule for R then there is a vector space decomposition

$$V = \bigoplus_{n \in \mathbf{Z}} V_n$$

such that the coefficient space of V_n is contained in R_n.

Before proving the proposition, we use it to prove Theorem 2. Let V be a polynomial representation of $F^\times$. Fix a basis $v_1, \ldots, v_d$ of v, and let $f_{ab} : F^\times \to F$ be the coordinate functions defined by equation (1). If $T : F^\times \to F$ denotes the identity function then the coordinate functions f_{ab} lie in $F[T]$. Now $F[T]$ is a coalgebra, with the coproduct Δ defined by

$$(\Delta f)(x, y) = f(xy).$$

Hence $\Delta T = T \otimes T$. Since the coproduct Δ commutes with the ordinary algebra product on $F[T]$, this determines Δ on all elements of $F[T]$. In particular, we have

$$\Delta T^r = T^r \otimes T^r \quad \text{for all } r \in \mathbf{N}_0$$

It follows that there is a coalgebra decomposition

$$F[T] = \bigoplus_{r \in \mathbf{N}_0} \langle T^r \rangle_F .$$

We may make V into a comodule for $F[T]$ via the map $\tau : V \to F[T] \otimes V$ defined by

$$\tau(v_b) = \sum_a f_{ab} \otimes v_a.$$

It therefore follows from the proposition that there is a comodule decomposition

$$V = \bigoplus_{r \in \mathbf{N}_0} V_r$$

such that the coefficient space of V_r is contained in $\langle T^r \rangle_F$. In other words, if $v \in V_r$ then $\rho(x)v = x^r v$ for all $x \in F^\times$. We have therefore succeeded in decomposing V into weight spaces.

Remark 1. It is an interesting exercise to repeat this argument but working with T_n rather than $T_1 = F^\times$. The coalgebra corresponding to T_n is an n-variable polynomial ring: the subspace of polynomials of a fixed degree is a subcoalgebra. Dualising the subcoalgebra consisting of polynomials of degree r gives a 'toroidal-Schur algebra'; this turns out to be semisimple and commutative with one primitive central idempotent for each possible weight space of degree r. Applying these idempotents to a polynomial representation decomposes it into weight spaces.

We may identify this algebra with the subalgebra of the usual Schur algebra $S(n, r)$ spanned by the idempotent elements ξ_α where α is a composition of r into at most n parts.

Remark 2. It follows from Proposition 3 that any polynomial representation of $\mathrm{GL}_n(F)$ decomposes as a direct sum of homogeneous representations, i.e. representations whose coordinate functions are polynomials of a fixed degree.

Proof of Proposition 3. Let

$$V_n = \{v \in V : \tau(v) \subseteq R_n \otimes V.\}$$

We first show that V_n is a subcomodule of V. The coaction τ respects the comultiplication Δ, so we have

$$(1_R \otimes \tau)\tau v = (\Delta \otimes 1_V)\tau v \quad \text{for all } v \in V.$$

Suppose that for a given $v \in V$ we have $\tau v = \sum_i f_i \otimes w_i$. Then it follows from the last equation that

$$\sum_i f_i \otimes \tau w_i = \sum_i \Delta f_i \otimes w_i.$$

Since $f_i \in R_n$, we have $\Delta f_i \in R_n$. Hence the right-hand side of the above belongs to $R_n \otimes R_n \otimes V$. Hence $\tau w_i \in R_n \otimes V$ for each i, and so $\tau v \in R_n \otimes V$, as required.

We now claim that $V = \oplus_n V_n$. Let $v \in V$. Since $R = \oplus_n R_n$, we may write τv as a sum of elements of $R_n \otimes V$, say

$$\tau v = \sum_n \sum_i f_i^n \otimes w_i^n \quad \text{where } f_i^n \in R_n \text{ and } w_i^n \in V \text{ for each } i.$$

Hence

$$\sum_n \sum_i \Delta f_i^n \otimes w_i^n = \sum_n \sum_i f_i^n \otimes \tau w_i^n.$$

It follows that $\tau w_i^n \in R_n \otimes V$ for each n and i, and so $w_i^n \in V_n$. Now, the comodule equivalent of the identify element in an algebra acting as the identity transformation is

$$(\epsilon \otimes \tau)v = v \quad \text{for all } v \in V.$$

Hence

$$v = \sum_n \sum_i \epsilon(f_i^n) w_i^n \in \sum_n V_n.$$

It only remains to show that the sum is direct. Let $v_i \in V_n$ and suppose that $\sum_i \lambda_i v_i = 0$. Applying τ we find that $\sum_i \lambda \tau v_i = 0$. But the τv_i are linearly independent (since $\tau v_i \in R_i \otimes V$), hence $\lambda_i = 0$ for all i. $\square$

3. Definition of the Schur functor

Fix $r \in \mathbf{N}$ and $n \in \mathbf{N}$ with $n \geq r$. Let V be a polynomial representation of $\mathrm{GL}_n(F)$, homogeneous of degree r. By Theorem 2, V decomposes as a direct sum of its weight spaces V_α where α is a composition of r with at most n parts. Let $\gamma = (1, 1, \ldots, 1, 0, \ldots, 0) \models r$ and let $\mathcal{F}V$ be the weight space V_γ. Thus

$$\mathcal{F}V = \{v \in V : tv = t_1 \ldots t_r v \quad \text{for all } t \in T_n.\}$$

Let $e_1, \ldots, e_n$ be the canonical basis of F^n. Let S_r denote the copy of the symmetric group of degree r consisting of those permutation matrices in $\mathrm{GL}_n(F)$ which permute $e_1, \ldots, e_r$ and fix the remaining basis elements.

Lemma 4. *If V be a polynomial representation of $\mathrm{GL}_n(F)$. then $\mathcal{F}V$ is invariant under the action of S_r.*

Proof. Let $v \in \mathcal{F}V$ and let $g \in S_r \subset \mathrm{GL}_n(F)$. Let $t \in T_n$. We have

$$t(gv) = g(g^{-1}tg)v = gt'v$$

where $t'_i = t_{ig^{-1}}$. Hence t' has the same set of entries as t in its first r diagonal positions, and $t'v = t_1 \ldots t_r v$. Therefore $gv \in \mathcal{F}V$, as required. $\square$

Suppose that V and W are polynomial representations of $\mathrm{GL}_n(F)$ and that $\theta : V \to W$ is a homomorphism. Let $v \in \mathcal{F}V$. We have

$$t\theta(v) = \theta(tv) = \theta(t_1 \ldots t_r v) = t_1 \ldots t_r \theta(v).$$

Hence θ restricts to a homomorphism $\mathcal{F}\theta : \mathcal{F}V \to \mathcal{F}W$. We have therefore shown that $\mathcal{F}$ is a functor from the category of homogeneous polynomial representations of $\mathrm{GL}_n(F)$ of degree r to the category of F-representations of S_r. This functor is known as the *Schur functor*.

Lemma 5. *The functor $\mathcal{F}$ is exact.*

Proof. Let

$$0 \to U \overset{\theta}{\hookrightarrow} V \overset{\phi}{\twoheadrightarrow} W \to 0$$

be a short exact sequence of polynomial representations of $\mathrm{GL}_n(F)$. Restriction is obviously exact, so the sequence

$$0 \to U\big\downarrow_{T_n} \overset{\theta}{\hookrightarrow} V\big\downarrow_{T_n} \overset{\phi}{\twoheadrightarrow} W\big\downarrow_{T_n} \to 0$$

is also exact. By Theorem 2, every polynomial representation of T_n is semisimple. Let $C \subseteq V$ be a T_n-complement to $\operatorname{im}\theta$. We may write the sequence

$$0 \to \mathcal{F}U \to \mathcal{F}V \to \mathcal{F}W \to 0$$

as

$$0 \to \mathcal{F}U \to \mathcal{F}U \oplus C \xrightarrow{\mathcal{F}\phi} \mathcal{F}W \to 0$$

where $\mathcal{F}\phi : C \to \mathcal{F}W$ is an isomorphism of vector spaces. Thus $\mathcal{F}$ is exact. □

Remark. It seems impossible to prove that $\mathcal{F}$ is right exact without knowing some version of Theorem 2.

Remark. The term 'Schur functor' is also used in another sense for a family of functors generalizing the symmetric and exterior powers of a vector space.

4. Appendix: some remarks on algebraic groups and comodules

4.1. **Coordinate rings.** Recall that the coordinate ring of $\mathrm{GL}_n(F)$ is

$$\mathcal{O}_{\mathrm{GL}_n(F)} = F[\det(X)^{-1}, X_{ij} : 1 \le i, j \le n]$$

where $\det(X)$ denotes the expected polynomial in the X_{ij}. A representation $\rho : \mathrm{GL}_n(F) \to \mathrm{GL}_d(F)$ is said to be *rational* if and only if the pullback map

$$\rho^\star : \mathcal{O}_{\mathrm{GL}_d(F)} \to \mathrm{Map}(\mathrm{GL}_n(F), F)$$

defined by

$$\rho^\star(h)g = h(\rho(g)) \quad \text{for } h \in \mathcal{O}_{\mathrm{GL}_d(F)}, g \in \mathrm{GL}_n(F)$$

has image contained in $\mathcal{O}_{\mathrm{GL}_n(F)}$. (Or equivalently, if and only if ρ is a morphism of algebraic groups.)

Let Y_{ab} be the coordinate functions on $\mathrm{GL}_d(F)$. We claim that a representation $\rho : \mathrm{GL}_n(F) \to \mathrm{GL}_d(F)$ is rational if and only if

$$\rho^\star(Y_{ab}) \in \mathcal{O}_{\mathrm{GL}_n(F)} \quad \text{for } 1 \le a, b \le d.$$

This condition is obviously necessary. Conversely, if it holds then, since $\rho^\star$ is a map of algebras, $\rho^\star(\det Y)$ lies in $\mathcal{O}_{\mathrm{GL}_n(F)}$. Now, $\rho^\star(\det Y)$ has inverse $\rho^\star(\det Y)^{-1}$, and it follows from the two lemmas below that the only invertible elements in $\mathcal{O}_{\mathrm{GL}_n(F)}$ are of the form $\lambda(\det X)^r$ for some $\lambda \in F$ and $r \in \mathbf{Z}$. Hence $\rho^\star(\det Y) \in \mathcal{O}_{\mathrm{GL}_n(F)}$. It follows that $\rho^\star\mathcal{O}_{\mathrm{GL}_d(F)} \subseteq \mathcal{O}_{\mathrm{GL}_n(F)}$.

Lemma 6. *The polynomial* $\det X \in F[X_{ij}]$ *is irreducible.*

Proof. Suppose that $\det X = fg$ where $f, g \in F[X_{ij}]$. For a fixed pair (i, j), we see that $\det X$ has degree 1 as a polynomial in X_{ij}. Hence each X_{ij} appears in at most one of f and g. Let A be the set of pairs (i, j) such that X_{ij} appears in f, and let B be the corresponding set for g. If (k, k) appears in A and (k, l) appears in B then $X_{kk}X_{kl}$ will appear in a monomial in fg, a contradiction. It follows that one of A and B is empty, and so either f or g is a unit. $\square$

Lemma 7. *Let* $A = F[Z_1, \ldots, Z_k]$ *be a polynomial ring of degree* k *and let* $f \in A$ *be irreducible. If* $g \in A[f^{-1}] = F[Z_1, \ldots, Z_k, f^{-1}]$ *is invertible then* $g = \lambda f^r$ *for some* $\lambda \in F$ *and* $r \in \mathbf{Z}$.

Proof. We may write

$$g^{-1} = \sum_{\alpha=0}^{M} f^{-\alpha} h_\alpha$$

for some polynomials $h_\alpha \in F[Z_1, \ldots, Z_k]$. Choose s sufficiently large that $gf^s \in F[Z_1, \ldots, Z_k]$. Multiplying through by gf^{M+s} we find that

$$f^{M+s} = gf^s \sum_{\alpha=0}^{m} f^{M-\alpha} h_\alpha.$$

This equation expresses an equality in the unique factorisation domain $F[Z_1, \ldots, Z_k]$, in which the only units are the elements of $F^\times$. By assumption f is irreducible. Hence we must have $gf^s = \lambda f^t$ for some $t \in \mathbf{N}$ and $\lambda \in F$, and so $g = \lambda f^{t-s}$. $\square$

We note that $\rho^\star(Y_{ab})$ is the function f_{ab} defined by (1). We can therefore characterise polynomial representations as those rational representations ρ such that $\rho^\star(F[X_{ab}]) \subseteq F[X_{ij}]$.

As an example of how coordinate rings can be used to prove results about representations, consider the following result.

Proposition 8. *Let* $\rho : \mathrm{GL}_n(F) \to \mathrm{GL}_d(F)$ *be a rational representation. Then there exists* $r \in \mathbf{Z}$ *such that* $\det \rho(g) = (\det g)^r$ *for all* $g \in \mathrm{GL}_n(F)$.

Proof. The pullback $\rho^\star(\det Y)$ is an invertible element of $\mathcal{O}_{\mathrm{GL}_d(F)}$. Hence, we may apply Lemma 6 and Lemma 7 to deduce that

$$\rho^\star(\det Y) = \lambda(\det X)^r$$

for some $r \in \mathbf{Z}$ and $\lambda \in F^\times$. Applying both sides to $g \in \mathrm{GL}_n(F)$ we get that $\det \rho(g) = \lambda(\det g)^r$ for all $g \in G$. Put $g = 1$ to get that $\lambda = 1$. $\square$

The r in the proposition can be determined by taking degrees. For example, applying the proposition to the example of $\mathrm{Sym}^2 E$ in §1.1 we get

$$\det \begin{pmatrix} \alpha^2 & \alpha\gamma & \gamma^2 \\ 2\alpha\beta & \alpha\delta + \beta\gamma & 2\gamma\delta \\ \beta^2 & \beta\delta & \delta^2 \end{pmatrix} = (\alpha\delta - \beta\gamma)^3.$$

This identity would be somewhat painful to prove by explicit calculation.

It is interesting to note that an analogous result holds for finite general linear groups. (But I cannot find a proof that will give both results at the same time.)

4.2. **Coalgebras and comodules.** Let G be an algebraic group. Let $\mu : G \times G$ be the multiplication map. Let $\Delta = \mu^\star$ be the pullback of μ, defined by

$$(\Delta f)(x, y) = f(xy) \quad \text{for } f \in \mathcal{O}_G,\ x, y \in G$$

By definition, the image of Δ inside $\mathrm{Map}(G \times G, F)$ is contained in $\mathcal{O}_G \otimes \mathcal{O}_{\mathcal{G}}$. In fact $\mathcal{O}_G$ becomes a *coalgebra* with coproduct Δ and counit $\epsilon : \mathcal{O}_G \to F$ defined by

$$\epsilon(f) = f(1_G) \quad \text{for } f \in \mathcal{O}_G.$$

It is useful to observe that the coproduct Δ commutes with the ordinary product on $\mathcal{O}_G$. That is,

$$(\Delta f)(\Delta h) = \Delta(fh) \quad \text{for all } f, h \in \mathcal{O}_G.$$

Thus $\mathcal{O}_G$ is in fact a bialgebra. The formal definition of a coalgebra is not terribly enlightening, but can easily be worked out by dualising the usual definition of an algebra.

References

[1] Green, J. A. Locally finite representations. *J. Algebra 41*, 1 (1976), 137–171.
[2] Green, J. A. *Polynomial representations of* GL_n, vol. 830 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1980.