Further exercises for a5 algebra

Questions 1 and 2 are there if you would like some revision of normal subgroups, but otherwise optional. If time is pressing, question 5 may also be regarded as optional. Question 3 is intended to clarify part of last week's sheet.

1. (Based on algebra moderations 2000 Q9). What are the elements and what is the definition of multiplication in the quotient G/N of a group G by a normal subgroup N? State carefully (without proof) the isomorphism theorem for groups.

Let G be the set of all 2×2 real matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a \neq 0$, and N the set of matrices in G with $a = 1$.

(a) Prove that G is a group under matrix multiplication.

(b) Define $\phi : G \to \mathbb{R}^\star$ by $\phi\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = a$. (Here $\mathbb{R}^\star$ is the multiplicative group of non-zero real numbers.) Prove that ϕ is a group homomorphism.

(c) Deduce that N is a normal subgroup of G and that G/N is isomorphic to $\mathbb{R}^\star$. Describe explicitly the elements of G/N.

(d) Are there any other normal subgroups of G?

2. Throught this question G is a finite group.

(a) Explain what it means to say that G is a simple group.

(b) Prove Cayley's theorem, that if G has order n, and $\rho : G \to S_n$ is the map sending each $g \in G$ to the permutation ρ_g it induces on G, then $G \cong \operatorname{im} \rho$.

(c) Suppose that the order of G is even. Show that G has an element of order 2. [*Hint: one approach is to consider a partitioning of G into subsets of the form $\{g, g^{-1}\}$.*]

(d) Suppose that G has order $2m$ where m is odd. Prove that if $t \in G$ has order 2 then ρ_t is an odd permutation of the elements of G. Hence prove that G has a normal subgroup of order m.

3. In this question we consider various groups of transformations of the plane, $\mathbb{R}^2$. Let $O_2(\mathbb{R})$ be the group of all distance-preserving linear maps from $\mathbb{R}^2$ to itself. Let $SO_2(\mathbb{R}) = \{T \in O_2(\mathbb{R}) : \det T = 1\ \}$. Let $T_a : \mathbb{R} \to \mathbb{R}$ be translation by $a \in \mathbb{R}^2$, i.e. $T_a(x) = a + x$. Let $T = \{T_a : a \in \mathbb{R}^2\}$ be the group of all translations. Let $E_2(\mathbb{R})$ be the group of isometries of the plane generated by $O_2(\mathbb{R})$ and $T_2(\mathbb{R})$.

(a) Show that if $x \in O_2(\mathbb{R})$ is represented by the matrix A with respect to the standard basis of $\mathbb{R}^2$ then $A^{tr}A = AA^{tr} = I$.

(b) Show that $SO_2(\mathbb{R})$ is a normal subgroup of $O_2(\mathbb{R})$. Describe geometrically the cosets of $SO_2(\mathbb{R})$ in $O_2(\mathbb{R})$.

(c) Prove that T is a normal subgroup of $E_2(\mathbb{R})$. Hence show that

$$E_2(\mathbb{R}) = \{T_a S : S \in O_2(\mathbb{R}), a \in \mathbb{R}^2\}.$$

Use this to give a rigorous proof that $\mathrm{Stab}_{E_2(\mathbb{R})}(0) = O_2(\mathbb{R})$.

4. Let G be a group of order p^a for some prime p. By considering the orbits in the action of G on itself by conjugacy, show that the centre of G has order at least p. [*The centre of a group G is $\{g \in G : xg = gx\ \forall x \in G\}$.*]

5. (Based on a3 algebra 2000 Q2). Let G be a non-trivial finite group of rotations of $\mathbb{R}^3$. Let P be the set of points of the unit sphere that are fixed by some non-identity element of G.

(a) Show that if $a \in P$ and $x \in G$ then the image of a under x lies in P.

(b) Prove that $|P|$ is an even integer with $2 \le |P| \le 2(|G| - 1)$.

(c) Let k be the number of orbits of G on P. Prove that $(k-2)\,|G| = |P| - 2$. [*Hint: use Burnside's lemma.*] Deduce that $2 \le k \le 3$.

(d) Show that if $|G|$ is odd then $|P| = k = 2$ and all non-identity elements of G have the same axis.

(e) Give an example where $k = 3$.