LIFTING SET/MULTISET DUALITY

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1. Introduction

Vandermonde's identity

$$\sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k} = \binom{a+b}{n}$$

can be proved very simply by interpreting each side as counting the number of k-subsets of $\{1,\ldots,a\} \cup \{a+1,\ldots,a+b\}$. Extending the definition of binomial coefficients by setting $\binom{x}{n} = x(x-1)\ldots(x-n+1)/n!$, for $x \in \mathbf{C}$ it holds replacing a and b with arbitrary $x, y \in \mathbf{C}$. For a quick proof of this, let $Q(x,y)$ denote the left-hand side and note that, for each $b \in \mathbf{N}_0$, the polynomial $Q(x,b) \in \mathbf{Q}[x]$ has roots at all $x \in \mathbf{N}_0$. Hence $Q(x,b) = 0 \in \mathbf{Q}[y]$ and hence $Q(x,y) \in \mathbf{Q}[x][y]$ is a polynomial in y having roots at all $y \in \mathbf{N}_0$, so is identically zero. In particular, if we take $D \in \mathbf{N}_0$ and set $x = -D$, $y = D$ we obtain

$$\sum_{k=0}^{n} \binom{-D}{k}\binom{D}{n-k} = 0 \tag{1.1}$$

for each $n \in \mathbf{N}$. Now writing $\left(\!\binom{D}{k}\!\right)$ for the number of k-multisubsets of a set of size D, and recalling (see for instance the previous blog post on the stars-and-bars identity) that

$$\binom{-D}{k}(-1)^k = \frac{D(D+1)\ldots(D+k-1)}{k!} = \binom{D+k-1}{k} = \left(\!\binom{D}{k}\!\right)$$

we obtain the reformulation

$$\sum_{k=0}^{n} (-1)^k \left(\!\binom{D}{k}\!\right)\binom{D}{n-k} = 0 \tag{1.2}$$

holding for $n \in \mathbf{N}$. (If $n < D$ then the sum may be started at $k = n - D$ with the earlier binomial coefficients vanishing by definition.) The special case $D = n$ gives the attractive $\sum_{k=0}^{n} (-1)^k \left(\!\binom{n}{k}\!\right)\binom{n}{k} = 0$.

There are of course many other ways to prove either (1.1) or (1.2). For instance, there is a nice one-line generating function proof using the Binomial Theorem by multiplying out $(1+x)^{-D}(1+x)^D$ and then taking the coefficient of x^n.

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The purpose of this note is to prove a 'high-altitude' lift of a generalisation of (1.2) by interpreting it as the alternating sum of dimensions in a long-exact sequence of representations of $\mathrm{GL}_D(F)$, where F is an arbitrary field. (This explains our choice of 'D' for 'dimension' above.) Having ascended to this altitude, we then descend in to easy stages to obtain first a symmetric function identity and second a q-binomial identity lifting (1.2).

2. Hook representations of general linear groups

Fix $D \in \mathbf{N}_0$ and let V be the natural D-dimensional representation of $\mathrm{GL}_D(F)$ with chosen basis $v_1, \ldots, v_D$. Let $t_{(i,j)}$ denote the entry in position (i,j) of a Young tableau t. Following the general construction in [1, §2], for each Young tableau t of shape $(k, 1^{n-k})$ with entries from $\{1, \ldots, D\}$ we define

$$f(t) = v_{t_{(1,1)}} v_{t_{(1,2)}} \ldots v_{t_{(1,k)}} \otimes v_{t_{(2,1)}} \otimes \ldots \otimes v_{t_{(n-k,1)}} \in \mathrm{Sym}^k V \otimes V^{\otimes(n-k)}.$$

Observe that $f(t)$ is zero if t has a repeated entry in its first column. When the entries are distinct, we define the GL-*polytabloid* $F(t)$ by

$$F(t) = \sum_u \varepsilon_u f(u)$$

where the sum is over all $(n-k+1)!$ distinct $(k, 1^{n-k})$-tableaux u obtained from t by permuting the entries in the first column, and ε_u is the sign of the permutation. The polynomial representation $\nabla^{(k,1^{n-k})}V$ of $\mathrm{GL}_D(\mathbf{C})$ is then the subspace of $\mathrm{Sym}^k V \otimes V^{\otimes(n-k)}$ with *standard basis*

$$\bigl\{F(t) : t \in \mathrm{SSYT}_{\leq D}(k, 1^{n-k})\bigr\}. \tag{2.1}$$

For a proof that this subspace is closed under the $\mathrm{GL}_D(F)$ action see [1, §2.4]. See [1, Remark 2.16] for a proof that our construction agrees with an earlier construction of James [3, Ch. 26] and so with [2, Ch. 4] in Green's lecture notes.

Example 2.1. For example, the GL-polytabloids

$$F\left(\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline j_1 \\ \cline{1-1} j_2 \\ \cline{1-1}\end{array}\right) = \begin{array}{l} v_{i_1}v_{i_2} \otimes v_{j_1} \otimes v_{j_2} - v_{i_1}v_{i_2} \otimes v_{j_2} \otimes v_{j_1} \\ + v_{j_1}v_{i_2} \otimes v_{j_2} \otimes v_{i_1} - v_{j_1}v_{i_2} \otimes v_{i_1} \otimes v_{j_2} \\ + v_{j_2}v_{i_2} \otimes v_{i_1} \otimes v_{j_1} - v_{j_2}v_{i_2} \otimes v_{j_1} \otimes v_{i_1}. \end{array}$$

for $i_1 \leq i_2$ and $i_1 < j_1 < j_2$ form a basis for $\nabla^{(2,1,1)}V$. Observe that swapping j_1 and j_2 gives a new (non-semistandard) tableau $t^\star$ for which $F(t^\star) = -F(t)$. Provided $\dim V \geq 3$, the GL-polytabloid $F(s)$ where s is as shown in the margin; is a highest weight vector of weight $(2,1,1)$ in $\nabla^{(2,1,1)}V$. When $F = \mathbf{C}$, this vector generates the full representation.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

More generally, a tableau of shape $(k, 1^{n-k})$ is semistandard if and only if it is of the form shown below

$$\begin{array}{|c|c|c|c|}\hline i_1 & i_2 & \cdots & i_k \\ \hline j_1 \\ \cline{1-1} j_2 \\ \cline{1-1} \\ \cline{1-1} j_{n-k} \\ \cline{1-1} \end{array}$$

with $i_1 \le i_2 \le \ldots \le i_k$ and $i_1 < j_1 < \ldots < j_{n-k}$.

As we saw in a special case in the example, each $F(t)$ is antisymmetric with respect to permutations permuting the entries $j_1, \ldots, j_{n-k}$ forming the 'leg' of the hook. Since $\bigwedge^r V$ is isomorphic as a $\mathrm{GL}(V)$-module to the subspace of $V^{\otimes r}$ of antisymmetric tensors (see for instance [2, §4.4, Example 2] or the section on exterior powers in [5] — note this is not obvious, and the analogous result for symmetric powers is false) the map

$$\nabla^{(k,1^{n-k})}V \to \mathrm{Sym}^k V \otimes \textstyle\bigwedge^{n-k} V$$

defined on the tableau t shown above by $F(t) \mapsto A(t)$ where

$$\begin{aligned} A(t) &= v_{i_1} v_{i_2} \ldots v_{i_k} \otimes v_{j_1} \wedge v_{j_2} \cdots \wedge v_{j_{n-k}} \\ &\qquad + \sum_{\alpha=1}^{n-k} (-1)^\alpha v_{j_\alpha} v_{i_2} \ldots v_{i_k} \otimes v_{i_1} \wedge v_{j_1} \wedge \cdots \wedge \widehat{v_{j_\alpha}} \wedge \cdots \wedge v_{j_{n-k}} \end{aligned}$$

is a homomorphism of $\mathrm{GL}(V)$-modules. (Here the hat denotes an omitted term.) It is clear from the standard basis of $\nabla^{(k,1^{n-k})}V$ in (2.1) that this map is injective. Therefore the image is a copy of $\nabla^{(k,1^{n-k})}V$ inside $\mathrm{Sym}_k V \otimes \bigwedge^{n-k} V$; we set

$$\boldsymbol{\nabla}^{(k,1^{n-k})}V = \big\langle A(t) : t \in \mathrm{SSYT}_{\le D}(n-k, 1^k) \big\rangle. \tag{2.2}$$

using bold face ∇ for this version of $\nabla^{(n-k,1^k)}V$. For instance, the element of $\mathrm{Sym}^2 V \otimes \bigwedge^2 V \in \boldsymbol{\nabla}^{(2,1,1)}V$ corresponding to the GL-polytabloid in the previous example is

$$A\left(\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline j_1 \\ \cline{1-1} j_2 \\ \cline{1-1}\end{array}\right) = v_{i_1} v_{i_2} \otimes v_{j_1} \wedge v_{j_2} - v_{j_1} v_{i_2} \otimes v_{i_1} \wedge v_{j_2} + v_{j_2} v_{i_2} \otimes v_{i_1} \wedge v_{j_1}. \tag{2.3}$$

3. A long exact sequence

3.1. **Setup.** As motivation for the following definition observe that (2.3) above is somewhat reminiscent of the image of the boundary map in simplicial homology. Throughout let $n \in \mathbf{N}$, $D \in \mathbf{N}_0$ and let V be a D-dimensional F-vector space.

Definition 3.1. For k such that $1 \leq n-k \leq D$, define

$$\delta_{n-k} : \mathrm{Sym}^k V \otimes \textstyle\bigwedge^{n-k} V \to \mathrm{Sym}^{k+1} V \otimes \textstyle\bigwedge^{n-k-1} V$$

by linear extension of

$$\delta_{n-k}(u_1 \cdots u_{n-k} \otimes w_1 \wedge \cdots \wedge w_{n-k}) = \sum_{\alpha=1}^{n-k} (-1)^\alpha u_1 \cdots u_{n-k} w_\alpha \otimes w_1 \wedge \cdots \wedge \widehat{w_\alpha} \wedge \cdots \wedge w_{n-k}$$

for $u_1, \ldots, u_{n-k}, w_1, \ldots, w_k \in V$.

For example, the element of $\mathrm{Sym}^2 V \otimes \bigwedge^2 V$ in (2.3) is $\delta_3(v_{i_2} \otimes v_{i_1} \wedge v_{j_1} \wedge v_{j_2})$, and so we immediately get $\mathrm{im}\, \delta_3 = \boldsymbol{\nabla}^{(2,1,1)} V \subseteq \mathrm{Sym}^2 V \otimes \bigwedge^2 V$. In the generic dimension case where $D \geq n$, the maps δ_r for $1 \leq r \leq D$ form the sequence

$$0 \to \textstyle\bigwedge^n V \xrightarrow{\delta_n} V \otimes \bigwedge^{n-1} V \xrightarrow{\delta_{n-1}} \cdots \\ \cdots \xrightarrow{\delta_{n-k+1}} \mathrm{Sym}^k V \otimes \bigwedge^{n-k} V \xrightarrow{\delta_{n-k}} \mathrm{Sym}^{k+1} V \otimes \bigwedge^{n-k-1} V \xrightarrow{\delta_{n-k-1}} \\ \cdots \xrightarrow{\delta_2} \mathrm{Sym}^{n-1} V \otimes V \xrightarrow{\delta_1} \mathrm{Sym}^n V \to 0. \quad (3.1)$$

Note that since $\bigwedge^D V$ is the determinant representation of $\mathrm{GL}(V)$, the start could be rewritten as $\det \xrightarrow{\delta_D} V \otimes \bigwedge^{D-1} V \xrightarrow{\delta_{D-1}}$. If instead $D < n$ then the terms with $\bigwedge^{n-r} V$ vanish for $r \leq n-D$ and the non-zero part of the sequence instead begins

$$0 \to \mathrm{Sym}^{n-D} V \otimes \textstyle\bigwedge^D V \xrightarrow{\delta_D} \mathrm{Sym}^{n-D-1} V \otimes \bigwedge^{D-1} V \xrightarrow{\delta_{D-1}} \cdots$$

where the first term may be rewritten as $\mathrm{Sym}^{n-D} V \otimes \det$. Our aim in this section is to show that this sequence is exact for general n and D.

Remark 3.2. When $D = n$, I cannot resist mentioning this high-brow proof that the sequence is exact, which works whenever F has characteristic zero. (Using results from [4] it can be generalized to any odd characteristic.) Applying the Schur functor to representations of the symmetric group S_D gives the sequence

$$0 \to \mathrm{sgn} \hookrightarrow \textstyle\bigwedge^{D-1} M \to \cdots \to \bigwedge^2 M \to M \to F \to 0$$

where M is the natural permutation module for S_D. Interpreting $\bigwedge^k M$ as the vector space of k-dimensional simplices of the solid $(D-1)$-dimensional simplex, this becomes the chain complex of a contractible connected space, augmented by a final map to F. Therefore the homology vanishes. Now use that in characteristic zero the Schur functor is invertible.

3.2. **Proof the sequence is long-exact.** For the general case, we first show that the sequence is a complex, i.e. the composition of any two consecutive differentials is zero. This is formally almost identical to the calculation for simplicial homology in Remark 3.2.

Lemma 3.3. *For any k such that $1 \le n-k < D$ the composition*

$$\delta_{n-k}\delta_{n-k+1} : \mathrm{Sym}^{k-1} \otimes \bigwedge^{n-k+1} V \to \mathrm{Sym}^{k+1} \otimes \bigwedge^{n-k-1} V$$

is the zero map.

Proof. The image under δ_{n-k+1} of the non-zero tensor $u_1 \cdots u_{k-1} \otimes w_1 \wedge \cdots \wedge w_{n-k+1}$ is

$$\sum_{\alpha=1}^{n-k+1} (-1)^\alpha u_1 \cdots u_{k-1} w_\alpha \otimes w_1 \wedge \cdots \wedge \widehat{w_\alpha} \wedge \cdots \wedge w_{n-k+1}. \tag{3.2}$$

Fix $\beta < \gamma$. Applying δ_{n-k} to the element above we see that $u_1 \cdots u_{m_1} w_\beta w_\gamma \otimes w_1 \wedge \cdots \wedge w_{n-k+1}$ appears once by taking $\alpha = \beta$ and then moving w_γ to the symmetric side of the tensor, and once by taking $\alpha = \gamma$ and then moving w_β to the symmetric side of the tensor. The signs are $(-1)^\beta(-1)^{\gamma-1}$ and $(-1)^\gamma(-1)^\beta$, respectively, which cancel. Summing over all pairs $1 \le \beta < \gamma < n-k+1$ accounts for all summands in the image under $\delta_{n-k}\delta_{n-k+1}$. The lemma follows. □

We also require the following combinatorial lemma. Given a multiset X and a set Y, with $\min X < \min Y$, let $t(X,Y)$ denote the unique semistandard tableau of shape $(|X|, 1^{|Y|})$ having first row entries X and first column entries $\{\min X\} \cup Y$.

Lemma 3.4. *For each k such that $1 \le n-k \le D$ we have*

$$\left|\mathrm{SSYT}_{\le D}(k, 1^{n-k})\right| + \left|\mathrm{SSYT}_{\le D}(k+1, 1^{n-k-1})\right| = \left(\!\!\binom{n}{k}\!\!\right)\binom{n}{k}.$$

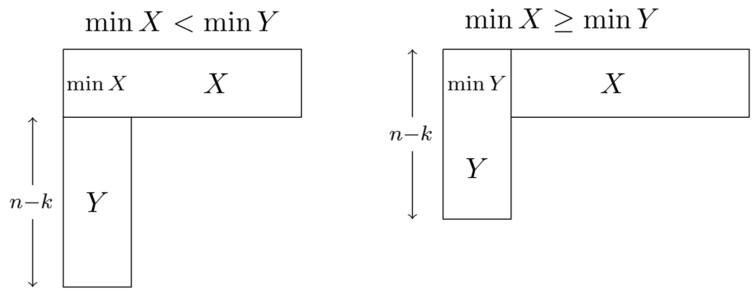
Proof. Let Ω be the set of all pairs (X,Y) where X is an k-multisubset of $\{1,\dots,D\}$ and Y is an $(n-k)$-subset of $\{1,\dots,D\}$. Given $(X,Y) \in \Omega$ *either* $\min X < \min Y$ and then

$$t(X,Y) \in \mathrm{SSYT}_{\le D}(k, 1^{n-k})$$

or $\min X \ge \min Y$ and then

$$t\big(X \cup \{\min Y\}, Y \backslash \{\min Y\}\big) \in \mathrm{SSYT}_{\le D}(k+1, 1^{n-k-1})$$

as shown in the diagram below.



We have therefore defined an injective map from Ω to $\mathrm{SSYT}_{\le D}(k, 1^{n-k}) \cup \mathrm{SSYT}_{\le D}(k+1, 1^{n-k-1})$. It is easily seen to be surjective. $\square$

Recall that $\nabla^{(k,1^{n-k})}V$ is the submodule of $\mathrm{Sym}^k V \otimes \bigwedge^{n-k} V$ defined in (2.2) as the span of all $A(t)$ for $t \in \mathrm{SSYT}_{\le D}(n-k, 1^k)$. By (3.2), we have

$$\delta_{n-k+1}(v_{i_2} \ldots v_{i_k} \otimes v_{i_1} \wedge v_{j_1} \wedge \cdots \wedge v_{n-k}) = A(t) \qquad (3.3)$$

where t is the tableau with first row entries $i_1, i_2, \ldots, i_k$ and first column entries $j_1, \ldots, j_{n-k}$ shown after Example 2.1.

Proposition 3.5. *The sequence* (3.5) *is exact and moreover*

$$\mathrm{im}\,\delta_{n-k+1} = \ker \delta_{n-k} = \nabla^{(k,1^{n-k})}V$$

for $1 \le n-k < D$.

Proof. Let k be such that $1 \le n-k \le D$ and consider the part

$$\xrightarrow{\delta_{n-k+1}} \mathrm{Sym}^k V \otimes \textstyle\bigwedge^{n-k} V \xrightarrow{\delta_{n-k}} \mathrm{Sym}^{k+1} V \otimes \bigwedge^{n-k-1} V \xrightarrow{\delta_{n-k-1}}$$

of the sequence, ignoring the left-most arrow if $n-k = D$. Let $1 \le i_1 \le \ldots \le i_k \le D$ and $j_1 < \ldots < j_{n-k} \le D$ with $i_1 < j_1$. By (3.3), $\mathrm{im}\,\delta_{n-k+1} = \nabla^{(k,1^{n-k})}V$. We make two deductions from this: first, by Lemma 3.3 that $\ker \delta_{n-k} \subseteq \mathrm{im}\,\delta_{n-k+1}$, we have

$$\ker \delta_{n-k} \subseteq \nabla^{(k,1^{n-k})}V. \qquad (3.4)$$

Second, by shifting k we get

$$\mathrm{im}\,\delta_{n-k} = \nabla^{(k+1,1^{n-k-1})}V.$$

By the rank-nullity theorem

$$\dim \ker \delta_{n-k} + \dim \mathrm{im}\,\delta_{n-k} = \dim\bigl(\mathrm{Sym}^k V \otimes \textstyle\bigwedge^{n-k} V\bigr) = \left(\!\binom{n}{k}\!\right)\binom{n}{k}.$$

Therefore, by Lemma 3.4, equality holds in (3.4) and we have $\ker \delta_{n-k} = \nabla^{(k,1^{n-k})}V$, which (as already used twice in this proof) is $\mathrm{im}\,\delta_{n-k+1}$. It only remains to check the ends: since $\delta_1 : \mathrm{Sym}^{n-1} V \otimes V \to \mathrm{Sym}^n V$ is the surjective multiplication map, the sequence is exact at the right-hand end. At the left-hand end, if $D \ge n$ then then it is clear that $\delta_n : \bigwedge^n V \to V \otimes \bigwedge^{n-1} V$ is injective. If instead $D < n$ then since $\bigwedge^{D+1} V = 0$, we have $\nabla^{(n-D,1^D)}(V) = 0$ and (3.4) implies that $\delta_D : \mathrm{Sym}^{n-D} V \otimes \bigwedge^D V \to$

$\operatorname{Sym}^{n-D+1} V \otimes \bigwedge^{D-1} V$ is injective. (The image is then $\nabla^{(n-D+1,1^{D-1})}V$.) The proposition follows. □

3.3. **Summary.** When $D \geq n$ we may therefore rewrite the long-exact sequence as

$$0 \to \textstyle\bigwedge^n V \xrightarrow{\delta_n} \begin{matrix}\nabla^{(2,1^{n-2})}V \\ \nabla^{(1^n)}V\end{matrix} \xrightarrow{\delta_{n-1}} \cdots$$

$$\cdots \xrightarrow{\delta_{n-k+1}} \begin{matrix}\nabla^{(m+1,1^{n-k-1})}V \\ \nabla^{(k,1^{n-k})}V\end{matrix} \xrightarrow{\delta_{n-k}} \begin{matrix}\nabla^{(m+2,1^{n-k-2})}V \\ \nabla^{(m+1,1^{n-k-1})}V\end{matrix} \xrightarrow{\delta_{n-k-1}}$$

$$\cdots \xrightarrow{\delta_2} \begin{matrix}\nabla^{(n)}V \\ \nabla^{(n-1,1)}V\end{matrix} \xrightarrow{\delta_1} \operatorname{Sym}^n V \to 0. \tag{3.5}$$

where we recall for ease of reference that the middle part is

$$\cdots \xrightarrow{\delta_{n-k+1}} \operatorname{Sym}^k V \otimes \textstyle\bigwedge^{n-k} V \xrightarrow{\delta_{n-k}} \operatorname{Sym}^{k+1} V \otimes \textstyle\bigwedge^{n-k-1} V \xrightarrow{\delta_{n-k-1}} .$$

If instead $D < n$ then the sequence begins

$$0 \to \operatorname{Sym}^{n-D} V \otimes \textstyle\bigwedge^D V \xrightarrow{\delta_D} \begin{matrix}\nabla^{(n-D+2,1^{D-2})}V \\ \nabla^{(n-D+1,1^{D-1})}V\end{matrix} \xrightarrow{\delta_{D-1}}$$

For example, if $D = 3$ and $n = 2$ we have

$$0 \to \textstyle\bigwedge^2 V \xrightarrow{\delta_2} V \otimes V \xrightarrow{\delta_1} \operatorname{Sym}^2 V \to 0$$

a sequence which is well known to split, with $V \otimes V = \operatorname{Sym}^2 V \oplus \bigwedge^2 V$, whenever F does not have characteristic 2. If instead $n = 4$ then we have

$$0 \to V \otimes \textstyle\bigwedge^3 V \xrightarrow{\delta_3} \operatorname{Sym}^2 V \otimes \textstyle\bigwedge^2 V \xrightarrow{\delta_2} \operatorname{Sym}^2 V \otimes \textstyle\bigwedge V \xrightarrow{\delta_2} \operatorname{Sym}^3 V \to 0$$

which can be rewritten as

$$0 \to \nabla^{(2,1,1)}V \xrightarrow{\delta_3} \begin{matrix}\nabla^{(3,1)}V \\ \nabla^{(2,1,1)}V\end{matrix} \xrightarrow{\delta_2} \begin{matrix}\nabla^{(4)}V \\ \nabla^{(3,1)}V\end{matrix} \xrightarrow{\delta_1} \operatorname{Sym}^4 V.$$

4. Specializations of the long-exact sequence

4.1. **Preliminaries.** We use the following lemma. Given a polynomial representation M of $\mathrm{GL}_D(F)$, let $f_M(x_1, \ldots, x_D)$ denote its character on the diagonal matrix $\operatorname{diag}(x_1, \ldots, x_D)$.

Lemma 4.1. *Let* $0 \to M_\ell \xrightarrow{\delta_\ell} M_{\ell-1} \xrightarrow{\delta_{\ell-1}} \cdots \xrightarrow{\delta_2} M_1 \xrightarrow{\delta_1} M_0 \xrightarrow{\delta_0} 0$ *be a long exact sequence of polynomial representations of* $\mathrm{GL}(V)$. *Then*

$$\sum_{k=0}^{\ell} (-1)^k f_{M_k} = 0.$$

Proof. By a generalisation of the rank-nullity theorem we have $f_{M_k} = f_{\ker \delta_k} + f_{\mathrm{im}\, \delta_k}$. By exactness, $\mathrm{im}\, \delta_k = \ker \delta_{k-1}$ for $1 \leq k \leq \ell$. Therefore the alternating sum is

$$f_{M_0} + \sum_{k=1}^{\ell} (-1)^k f_{\ker \delta_k} + \sum_{k=1}^{\ell} (-1)^k f_{\ker \delta_{k-1}} = f_{M_0} - f_{\ker \delta_0} = 0$$

as required. □

Since it may be useful in generalizations of this note, we mention that extending an arbitrary long exact sequence

$$M_\ell \xrightarrow{\delta_\ell} M_{\ell-1} \to \cdots \to M_1 \xrightarrow{\delta_0} M_0$$

by $\ker \delta_\ell$ at the start and $\mathrm{coker}\, \delta_0 = M_0 / \mathrm{im}\, \delta_0$ at the end and then applying Lemma 4.1 gives the more general

$$\sum_{k=0}^{\ell} (-1)^k f_{M_k} = (-1)^k \dim \ker \delta_\ell + f_{M_0} - f_{\mathrm{im}\, \delta_0} \tag{4.1}$$

which can be reformulated for any long exact sequence in an abelian category $\mathcal{A}$, by replacing characters with isomorphism classes in the Grothendieck group $K_0(\mathcal{A})$. An important special case is the generalized Euler's formula $F - E + V = 2 - 2g$ for a triangulation of an orientable surface of genus g.

4.2. **Symmetric functions.** Recall that h_k is the complete homogeneous symmetric function of degree k and e_k is the elementary symmetric function of degree k. Taking V of dimension D as usual, we have $h_k(x_1, \ldots, x_D) = f_{\mathrm{Sym}^k V}(x_1, \ldots, x_D)$ and $e_k(x_1, \ldots, x_D) = f_{\wedge^k V}(x_1, \ldots, x_k)$.

Corollary 4.2. *For $n \in \mathbf{N}$ and $D \in \mathbf{N}_0$ we have*

$$\sum_k (-1)^k h_k(x_1, \ldots, x_D) e_{n-k}(x_1, \ldots, x_D) = 0.$$

Proof. Observe that $e_{n-k}(x_1, \ldots, x_D) = 0$ unless $0 \leq n - k \leq D$, or equivalently, $n - D \leq m \leq D$. From $h_k(x_1, \ldots, x_D)$ we require just that $m \geq 0$. The sum is therefore over k such that $\max(0, n - D) \leq m \leq D$. The result now follows by applying Lemma 4.1 to Proposition 3.5. □

It is a nice exercise to give an alternative proof of this corollary using Young's rule or Pieri's rule to express the product $h_k e_{n-k}$ as the sum $s_{(n-k,1^k)} + s_{(n-k+1,1^{k-1})}$ and then cancelling terms, almost exactly as in the proof of Lemma 4.1.

4.3. **q-binomial coefficients.** Define the q-number $[k]_q$ by $[k]_q = (q^k - 1)/(q-1) = 1 + q + \cdots + q^{k-1}$, the q-factorial by $[k]_q! = [k]_q[k-1]_q \ldots [1]_q$ and the q-binomial coeficient

$$\begin{bmatrix} D \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

We mention that one combinatorial interpretation of the q-binomial coefficients is that $q^{k(k-1)/2}\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = \sum_{X \subseteq \{0,1,\ldots,n-1\}} q^{\sum X}$ and so up to a power of q, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ is the principal specialization of e_k:

$$q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} = e_k(1, q, \ldots, q^{n-1}). \tag{4.2}$$

By the bijection between k-multisubsets of $\{0, 1, \ldots, n-1\}$ and k-subsets of $\{0, 1, \ldots, n+k-2\}$ defined by adding $j-1$ to the jth smallest element, we obtain the dual identity

$$\begin{bmatrix} n+k-1 \\ k \end{bmatrix} = h_k(1, q, \ldots, q^{n-1}). \tag{4.3}$$

In particular the q-binomial coeficients are polynomials in q. (At least this is the case for our definition: there is an alternative definition, very useful for quantum groups, where q-binomial coefficients are Laurent polynomials.) By convention $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right] = 0$ if k is negative, and it vanishes by definition if $k > n$.

Corollary 4.3. *We have*

$$\sum_r (-1)^r q^{r(r-1)/2} \begin{bmatrix} n-r+D-1 \\ n-r \end{bmatrix}_q \begin{bmatrix} D \\ r \end{bmatrix}_q = 0.$$

Proof. Apply (4.2) and (4.3) to Corollary 4.2 and then change the summation variable by setting r equal to $n-k$. □

The non-zero terms in the sum come from r such that $0 \le r \le \min(n, D)$. Again it is a very instructive exercise to find a combinatorial proof of this identity.

Corollary 4.4. *We have*

$$\sum_r (-1)^r \left(\!\!\binom{D}{n-r}\!\!\right) \binom{D}{r} = 0.$$

Proof. Take $q = 1$ in Corollary 4.3. □

In particular, by changing summation variables once again, we obtain (1.2), our original motivation.

References

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