WEAK LUMPING OF LEFT-INVARIANT RANDOM WALKS ON LEFT COSETS OF FINITE GROUPS

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ABSTRACT. Let G be a finite group and let H be a subgroup of G. The *left-invariant random walk* driven by a probability measure w on G is the Markov chain in which from any state $x \in G$, the probability of stepping to $xg \in G$ is $w(g)$. The initial state is chosen randomly according to a given distribution. The walk is said to *lump weakly on left cosets* if the induced process on G/H is a time-homogeneous Markov chain. We characterise all the initial distributions and weights w such that the walk is irreducible and lumps weakly on left cosets, and determine all the possible transition matrices of the induced Markov chain. In the case where H is abelian we refine our main results to give a necessary and sufficient condition for weak lumping by an explicit system of linear equations on w, organized by the double cosets HxH. As an application we consider shuffles of a deck of n cards such that repeated observations of the top card form a Markov chain. Such shuffles include the random-to-top shuffle, and also, when the deck is started in a uniform random order, the top-to-random shuffle. We give a further family of examples in which our full theory of weak lumping is needed to verify that the top card sequence is Markov.

CONTENTS

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1. Introduction

Let G be a finite group. We define a *weight* to be a function on G taking non-negative real values, at least one of which is positive. Thinking of w as a measure on G, given any subset K of G, we write $w(K)$ for $\sum_{g\in K} w(g)$. The *left-invariant random walk* on G driven by the weight w is the time-homogeneous G-valued Markov chain $X = (X_0, X_1, X_2, \dots)$ with transition probabilities

$$\mathbb{P}[X_{t+1} = xg \mid X_t = x] = \frac{w(g)}{w(G)}.$$

The starting value X_0 may be either deterministic or random. Let H be a subgroup of G. (Throughout, H and G have these meanings.) Many natural questions concern the induced random process $(X_0H, X_1H, \ldots)$ taking values in the set G/H of left cosets of H in G. For instance, if $G = \mathrm{Sym}_n$ and $H = \mathrm{Sym}_{n-1}$, then the left-invariant random walk on G models a sequence of random shuffles of a deck of n cards, and the induced process on G/H models the sequence of cards appearing as the top card in the deck after each shuffle. This is the setting of the extended example of our main results in §1.2, where we give further justification for our focus on left cosets.

In our stochastic setting, it is natural to start the left-invariant random walk X at a *random* starting point X_0, distributed according to a chosen probability distribution α on G. We denote the resulting Markov chain by $\mathrm{MC}(\alpha, w)$. By the main theorem of [8], the induced process on G/H is a time-homogeneous Markov chain *for every initial distribution* α if and only if, for every double coset HxH of H in G, $w(gH)$ is constant over all $gH \subseteq HxH$. (See §4 for background on double cosets.) In this case we say that the random walk on G *lumps strongly* on the left cosets of H.

Definition 1.1. Let w be a weight.

(a) We say that the left-invariant random walk driven by w *lumps weakly to* G/H *when started at the distribution* α if the induced process $(X_tH)_{t\geq 0}$ taking values in G/H is a time-homogeneous Markov chain, when X_0 is distributed according to α.

(b) We say the left-invariant random walk driven by w *lumps weakly to* G/H if it lumps weakly to G/H for *some* initial distribution α.

More briefly, if (a) holds we say that $\mathrm{MC}(\alpha, w)$ *lumps weakly* to G/H and if (b) holds then w *lumps weakly* or is *weakly lumping*. We shall see in this article that weak lumping occurs much more generally than strong lumping, and has a much deeper theory.

We concentrate on the case when the weight w is *irreducible*; that is, the support of w generates G, or equivalently, the random walk X is an irreducible Markov chain, with the uniform distribution as its unique invariant distribution. We justify this choice in §1.3: we plan to study the reducible case further in a sequel to this paper.

The main contributions of this paper are:

(1) A complete algebraic description of the set of irreducible weights w such that the left-invariant random walk on G driven by w lumps

weakly to G/H when started at the uniform distribution on G (Theorem 1.2 and Corollary 1.6), together with an algorithm that determines whether this holds for any given weight w;

(2) For any given irreducible weight w, a complete algebraic description of the set of probability distributions α on G such that $\mathrm{MC}(\alpha, w)$ lumps weakly to G/H, and a procedure to compute this set (Theorem 1.7).

A key idea is to interpret a weight w on the group G as the element $\sum_{g\in G} w(g)g$ of the complex group algebra $\mathbb{C}[G]$. As we show in Lemma 3.1, steps in the Markov chain then correspond to multiplication by w in $\mathbb{C}[G]$, and so we are able to bring representation theory to bear on our problem. To avoid difficulties caused by working over a non-algebraically closed field, we must work over $\mathbb{C}$ even though weights are real valued.

When read with §2 on the preliminaries we need from Markov chain theory, and §3 on the algebraic preliminaries, we hope this paper will be found accessible to a broad readership.

1.1. **Main results.** Given a non-empty subset K of G, the element of $\mathbb{C}[G]$ corresponding to the uniform distribution on K is

$$\eta_K = \frac{1}{|K|} \sum_{g \in K} g.$$

Recall that $e \in \mathbb{C}[G]$ is an *idempotent* if $e^2 = e$. For example, η_K is an idempotent whenever K is a subgroup of G. Let $E(H)$ be the set of idempotents of $\mathbb{C}[H]$ and let $E^\bullet(H)$ be the subset of idempotents e such that $\eta_H e = \eta_H$. Recall that $(X_t)_{t\geq 0}$ denotes the Markov chain $\mathrm{MC}(\alpha, w)$.

Theorem 1.2 (Characterisation of weak lumping in terms of idempotents)**.** *Let w be an irreducible weight on G and let α be a distribution on G, both thought as elements of $\mathbb{C}[G]$. Then $\mathrm{MC}(\alpha, w)$ lumps weakly to G/H if and only if there exists an idempotent $e \in E^\bullet(H)$ such that*

(i) $\alpha \in \mathbb{C}[G]e$,

(ii) $ew(1 - e) = 0$,

(iii) $(e - \eta_H)w\eta_H = 0$.

In this case, for any $t \geq 0$, the conditional distribution of X_t given the sequence of cosets X_0H, ..., X_tH always belongs to $\mathbb{C}[G]e$.

Note that $\mathbb{C}[G]e$ is a left ideal of $\mathbb{C}[G]$. It follows from the final part of Theorem 1.2, by averaging over all sequences of cosets $X_0H, \ldots, X_tH$, that the distribution of X_t is in $\mathbb{C}[G]e$ for every time t. These observations motivate the following definition.

Definition 1.3. Let L be a left ideal of $\mathbb{C}[G]$ of the form $L = \mathbb{C}[G]e$ for $e \in E^\bullet(H)$. We say that the left-invariant random walk driven by an irreducible weight *w lumps weakly to G/H with stable ideal L* if $ew(1 - e) = 0$ and $(e - \eta_H)w\eta_H = 0$.

There may be more than one $e \in E^\bullet(H)$ such that $L = \mathbb{C}[G]e$. However, we shall show (see Lemma 5.14) that the conditions $ew(1 - e) = 0$ and $(e - \eta_H)w\eta_H = 0$ in Definition 1.3 either hold for all choices of e or none, since they are equivalent to $Lw \subseteq L$ and $L(1 - \eta_H)w\eta_H = 0$ respectively.

We may write 'w lumps stably for L' as shorthand for Definition 1.3. It follows from Theorem 1.2 that if w lumps stably for L then

(a) $X = \mathrm{MC}(\alpha, w)$ lumps weakly to G/H for all initial distributions $\alpha \in L$, *and*

(b) for any initial distribution $\alpha \in L$ and all t, L always contains the conditional distribution of X_t given $X_0H, \ldots, X_tH$.

The card shuffling example in §1.2 demonstrates that stable lumping is interesting even in cases where the left-invariant random walk lumps strongly. Thus for each $e \in E^\bullet(H)$, Theorem 1.2 gives a necessary and sufficient condition for the left-invariant random walk to lump stably for $\mathbb{C}[G]e$. In Corollary 8.3 we refine Theorem 1.2 to show that, in the irreducible case, all weakly lumping weights can be obtained by considering real idempotents in $E^\bullet(H) \cap \mathbb{R}[H]$.

Remark 1.4. A left ideal L of $\mathbb{C}[G]$ may be expressed as $L = \mathbb{C}[G]e$ for some idempotent $e \in E(H)$ if and only if L decomposes as a direct sum of its projections to the subspaces $b\mathbb{C}[H]$ of $\mathbb{C}[G]$, i.e. $\mathbb{C}[G]e = \bigoplus_{b \in G/H} b\mathbb{C}[H]e$. (Here and throughout, the notation $b \in G/H$ means that b varies over a set of representatives for the left cosets bH of H in G.) This decomposition shows that, as a left $\mathbb{C}[G]$-ideal, $\mathbb{C}[G]e$ is isomorphic to the *induced ideal* $(\mathbb{C}[H]e)\big\uparrow_H^G$. (See Definitions 3.12 and 3.15 for the definition of induction and induced ideals.) This connection between weak lumping and induced ideals is a recurring theme in this work and critical to the proof of Theorem 1.2.

In Definition 5.11 below we define a *Gurvits–Ledoux ideal* for an irreducible weight w to be an induced left ideal L of $\mathbb{C}[G]$ containing η_G and such that $Lw \subseteq L$. We show in Proposition 5.12 that the left-invariant random walk driven by w lumps stably for the Gurvits–Ledoux ideal L if and only if $L(1 - \eta_H)w \subseteq L(1 - \eta_H)$. This leads to a practical computational test for weak lumping, starting at a distribution. By Definition 5.11, $L_{\alpha,w}$ is the intersection of all Gurvits–Ledoux ideals containing α.

Corollary 1.5. *Let w be an irreducible weight. The Markov chain* $\mathrm{MC}(\alpha, w)$ *lumps weakly to G/H if and only if $L_{\alpha,w}(1 - \eta_H) \subseteq L_{\alpha,w}$.*

In §6, we provide a practical computational procedure to compute $L_{\alpha,w}$, and in particular to compute $L_{\eta_G,w}$ for any given weight w. It is an important feature of this test that the computation of the left ideal $L_{\alpha,w}$ and the test of whether $L_{\alpha,w}(1 - \eta_H) \subseteq L_{\alpha,w}$ may be performed almost entirely within $\mathbb{C}[H]$, making it more efficient than a direct application of the Gurvits–Ledoux criterion (Theorem 2.6) when H is much smaller than G.

As a corollary, we obtain a practical test for weak lumping, in the wider sense of Definition 1.1(b). Set $L_w = L_{\eta_G,w}$.

Corollary 1.6 (Weak lumping test for a weight)**.** *Let w be an irreducible weight. The following are equivalent:*

(i) *The left-invariant random walk driven by w lumps weakly to G/H;*

(ii) $\mathrm{MC}(\eta_G, w)$ *lumps weakly to G/H;*

(iii) $L_w(1 - \eta_H)w\eta_H = 0$*;*

(iv) *The left-invariant random walk driven by w lumps weakly to G/H with stable ideal L_w.*

Dual to the minimal ideal L_w in the previous corollary, we show in §5.3 that for each irreducible weight w there exists a maximal Gurvits–Ledoux ideal L satisfying $L(1-\eta_H)w\eta_H = 0$. We denote this ideal J_w. We provide a practical computational procedure to compute J_w and use it to describe the set of initial probability distributions α for which $\mathrm{MC}(\alpha, w)$ lumps weakly to G/H.

Theorem 1.7 (Weak lumping test for an initial distribution)**.** *Let $w \in \mathbb{C}[G]$ be an irreducible weakly lumping weight. For each distribution α on G, the Markov chain $\mathrm{MC}(\alpha, w)$ lumps weakly to G/H if and only if $\alpha \in J_w$.*

Returning to Theorem 1.2, given $e \in E^\bullet(H)$, let

$$\Theta(e) = \big\{ w \in \mathbb{C}[G] : ew(1-e) = 0, (e - \eta_H)w\eta_H = 0 \big\} \tag{1.1}$$

and let

$$\Theta = \bigcup_{e \in E^\bullet(H)} \Theta(e).$$

Theorem 1.2 implies that a weight w lumps weakly in the sense of Definition 1.1(b) if and only if $w \in \Theta$; moreover in this case, as noted above, w lumps weakly to G/H with stable ideal $\mathbb{C}[G]e$, in the sense of Definition 1.3. Equivalently, the set of weakly lumping irreducible weights is $\Gamma \cap \Delta \cap \Theta$ where $\Delta \subseteq \mathbb{R}[G]$ is the simplex of probability distributions and Γ is the set of elements of $\mathbb{C}[G]$ whose support is not contained in any proper subgroup of G, i.e.

$$\Gamma = \mathbb{C}[G] \setminus \bigcup_{K \lneq G} \mathbb{C}[K].$$

Remarkably, as we show in Lemma 7.1, $\Theta(e)$ is a subalgebra of $\mathbb{C}[G]$; that is, $\Theta(e)$ is a vector subspace of $\mathbb{C}[G]$ closed under multiplication.

We have an interesting characterisation of when the union defining Θ is irredundant. To state it, we require Definition 7.16: if the restriction to the subgroup H of the permutation character of G acting on the cosets of H contains every irreducible character of H, then we say that H has *full induction restriction.* For instance, H has full induction restriction whenever there is a double coset HxH of the maximum possible size $|H|^2$, or equivalently, whenever the permutation group of G acting on the left cosets of H has a base (see [11, §4.13]) of size 2.

Proposition 1.8. *The subgroup H of G has full induction restriction if and only if the union defining Θ is irredundant, in the sense that no subalgebra $\Theta(e)$ is contained in another.*

We remark that for each choice of $e \in E^\bullet(H)$, the conditions in (1.1) give a finite system of *linear* equations that define $\Theta(e)$. These equations can be re-expressed so that each equation refers only to values $w(g)$ for g in a fixed double coset HxH. This is made explicit in Corollary 1.12 and (12.2) and makes Theorem 1.2 a computationally effective result. We also highlight Proposition 12.6, which states that the number of equations in an irredundant system of equations for the condition $ew(1-e) = 0$ on $\mathbb{C}[HxH]$ is

$$\big\langle \chi_{\mathbb{C}[H]e}\big\downarrow_{H\cap xHx^{-1}}, \big(\chi^{x^{-1}}_{\mathbb{C}[H](1-e)}\big)\big\downarrow_{H\cap xHx^{-1}} \big\rangle.$$

Here $\chi_{\mathbb{C}[H]e}$ is the character of the left $\mathbb{C}[H]$-ideal $\mathbb{C}[H]e$, the downwards arrow denotes restriction, and the inner product is as defined in (1.2) below taking $G = H \cap xHx^{-1}$; see §3 for the remaining notation. This gives a good flavour of how representation theory leads to results on our probabilistic questions.

Let $\star$ denote the algebra anti-involution on $\mathbb{C}[G]$ defined, for $x \in \mathbb{C}[G]$, by $x^\star = \sum_{g \in G} \overline{x(g)} g^{-1}$. There is a beautiful duality between the left-invariant random walk driven by a weight w and its time-reversal, which is the left-invariant random walk driven by the weight $w^\star$. (Note that since w is real-valued, $w^\star = \sum_{g \in G} w(g) g^{-1}$.)

Theorem 1.9 (Time reversal)**.** *Let $e \in E^\bullet(H)$ and let $w \in \mathbb{C}[G]$ be a weight. The left-invariant random walk on G driven by w lumps weakly to G/H with stable ideal $\mathbb{C}[G]e$ if and only if the left-invariant random walk on G driven by $w^\star$ lumps weakly to G/H with stable ideal $\mathbb{C}[G](1 - e^\star + \eta_H)$.*

We have seen that strong lumping is a sufficient condition for weak lumping. There is another commonly used sufficient condition for weak lumping, called *exact lumping* (see Definition 2.19). In our setting it corresponds to taking $e = \eta_H$ in Theorem 1.2. Applying Theorem 1.9 to the characterisation in [8] of strong lumping, stated as (i) in the corollary below, we obtain a simple criterion for exact lumping. The weight w in the following corollary may be reducible.

Corollary 1.10. *The left-invariant random walk driven by a weight w*

(i) *lumps strongly to G/H if and only if $w(gH)$ is constant for left cosets gH in the same double coset;*

(ii) *lumps exactly to G/H if and only if $w(Hg)$ is constant for right cosets Hg in the same double coset.*

We remark that Propositions 2.25 and 2.26 give attractive reinterpretations of strong and exact lumping using conditional independence that may be applied to this corollary. In Proposition 11.1 we show that this corollary describes the two extreme cases of a family of results on weak lumping to G/H indexed by the subgroups of H: strong lumping is the case of the trivial subgroup, and exact lumping is the case where the subgroup is H itself.

It is natural to ask for the possible transition matrices of the lumped process when the left-invariant random walk on G lumps weakly to G/H. This is addressed by our next main theorem. See §9.1 for the definition of the orbital matrices M_{HxH}. To orient the more expert reader we remark that the algebra $\eta_H \mathbb{C}[G] \eta_H$ in (iii) is isomorphic to the Hecke algebra of H-bi-invariant functions on the double coset space $H \backslash G / H$. The weights appearing in conditions (i), (ii) and (iv) below may be reducible.

Theorem 1.11. *Let Q be a stochastic matrix with rows and columns indexed by G/H. The following are equivalent:*

(i) *there is a weight w on G such that* $\mathrm{MC}(\eta_G, w)$ *lumps weakly to G/H and the lumped chain has transition matrix Q;*

(ii) *Q is the transition matrix of the induced random walk on G/H driven by a weight w in $\eta_H \mathbb{C}[G] \eta_H$.*

(iii) *Q satisfies $Q_{(gH,g'H)} = Q_{(kgH,kg'H)}$ for all $g, g', k \in G$;*
(iv) *Q is the transition matrix of the induced random walk on G/H driven by a weight satisfying the conditions in Corollary 1.10 to lump both strongly and exactly on G/H.*

Our final main result is on the case when H is abelian. In this case the set $E^\bullet(H)$ is finite, and so we can make Theorem 1.2 very explicit. Let $\widehat{H}$ denote the set of irreducible linear characters of H and let $\mathbb{1}_H \in \widehat{H}$ denote the trivial character. Let $e_\phi = |H|^{-1} \sum_{h\in H} \phi(h^{-1})h$ denote the centrally primitive idempotent in $\mathbb{C}[H]$ corresponding to the irreducible character $\phi \in \widehat{H}$. Let $\langle -, - \rangle$ denote the G-invariant inner product on $\mathbb{C}[G]$ defined by

$$\Big\langle \sum_{g\in G} \alpha(g)g, \sum_{k\in G} \beta(k)k \Big\rangle = \frac{1}{|G|} \sum_{g\in G} \overline{\alpha}(g)\beta(g). \tag{1.2}$$

Given a subspace V of $\mathbb{C}[G]$, let $V^\perp = \{w \in \mathbb{C}[G] : \langle v, w\rangle = 0 \text{ for all } v \in V\}$.

Corollary 1.12. *Let D be a set of double coset representatives for $H\backslash G/H$. The left-invariant random walk on G driven by an irreducible weight w lumps weakly on the left cosets of H if and only if there exists a subset $P \subseteq \widehat{H}$ containing $\mathbb{1}_H$ such that for all $x \in D$ we have $w \in \bigcap_{x\in D} W_x^\perp$, where*

$$W_x = \big\langle e_\beta x e_\gamma : \ \beta \in P,\ \gamma \in (\widehat{H}\backslash P) \cup \{\mathbb{1}_H\},\ (\beta,\gamma) \neq (\mathbb{1}_H, \mathbb{1}_H)\big\rangle.$$

We emphasise that the subspace W_x of $\mathbb{C}[G]$ appearing above is contained in the double coset HxH and so each perpendicular space in the intersection supplies $\dim W_x$ independent equations that must be satisfied by the values $w(g)$ for $g \in HxH$. This is illustrated in our final example in §13.2.

1.2. **Extended example: shuffles that frustrate card counters.** This extended example includes §1.2.4 where we give an infinite family of shuffles in the symmetric groups Sym_n that lump weakly but not lump strongly or exactly.

To get started, consider a deck of four cards, the Queen on top in position 1, then the Jack in position 2, then the Ace in position 3 and finally the King in position 4 at the bottom, as shown in the margin. (We pick this non-obvious order to distinguish more forcefully between positions and card values.) Permutations in Sym_4 act on the deck by permuting positions: if card C is in position j then, after the shuffle $g \in \mathrm{Sym}_4$, card C is in position jg. (Note that we act on the right, and so permutations compose left-to-right: $j(gh) = (jg)h$.)

Q	1
J	2
A	3
K	4

To put this into our algebraic setting, let $G = \mathrm{Sym}_4$ and let $H = \mathrm{Sym}_{\{2,3,4\}}$ be the subgroup of permutations that fix the top card. The left cosets G/H are $H, (1,2)H, (1,3)H, (1,4)H$; the left coset $(1,k)H$ is precisely those permutations moving the card in position k to position 1. The induced process on left cosets therefore models the changing *value* of the top card: 'what a card-counter sees'. A card-counter acquires no useful information by memorizing the past history of values of the top card if and only if this induced process is a Markov chain. (We shall see an explicit example of this shortly.) Dually, the right cosets $H\backslash G$ are $H, H(1,2), H(1,3), H(1,4)$; the right-coset $H(1,k)$ is precisely those permutations moving the card in position 1 to position k. As noted earlier by Pang in [28, Example 2.9] the induced process on right-cosets models the changing *position* of the initial top card, here the Queen: 'follow the lady'.

Lumping on right cosets. Since the left-invariant random walk (defined for general G and H) is left-invariant, we have in particular

$$\mathbb{P}[X_t = g' | X_{t-1} = g] = \mathbb{P}[X_t = hg' | X_{t-1} = hg]$$

for all g, $g' \in G$ and $h \in H$. Hence $\mathbb{P}[HX_t = Hg' | X_{t-1} = hg]$ is constant as h varies, and any left-invariant random walk lumps strongly on right cosets. In particular it is always a Markov chain. In our shuffling setup, both claims are obvious: the position of the Queen after a shuffle depends only on its position before the shuffle. Since strong lumping implies weak lumping, it follows that the left-invariant random walk lumps weakly on right cosets for *any* weight w and initial distribution. Similarly, if X_{t-1} is distributed uniformly on Hg, then the distribution of X_t conditioned on $HX_t = Hg'$ is uniform on Hg'. Hence the left-invariant random walk lumps exactly on right cosets, in the sense of Definition 2.19, for *any* weight w. In our shuffling setup, this says (for instance) that if we know the Ace, King and Jack are uniformly distributed on the three positions known not to have the Queen, then the same is true after the deck is shuffled. These relatively easy observations contrast with the much deeper theory for left cosets lumping in this article.

Lumping on left cosets. Since H has two orbits on $\{1, 2, 3, 4\}$, there are two double cosets, H itself, corresponding to the orbit $1H = \{1\}$, and $H(1, 2)H$ corresponding to the orbit $1(1, 2)H = \{2, 3, 4\}$. The larger double coset is shown in Figure 1.

1.2.1. Strong lumping. By Corollary 1.10(i), the left-invariant random walk driven by a weight w lumps strongly to G/H if and only $w(gH)$ is constant for $g \in H(1, 2)H$, or equivalently, if and only if $w\big((1, 2)H\big) = w\big((1, 3)H\big) = w\big((1, 4)H\big)$. (Note the subgroup H is itself a double coset and gives no restriction.) In Figure 1, the condition is that each row has equal weight. In particular, this holds whenever the non-identity part of w is uniformly distributed on any fixed right coset of H. Taking one element of the right coset $H(1, 2)$ in each left coset of H, together with the identity, we obtain the *random-to-top* shuffle $\frac{1}{4}\mathrm{id} + \frac{1}{4}(1, 2) + \frac{1}{4}(1, 2, 3) + \frac{1}{4}(1, 2, 3, 4)$.

1.2.2. Exact lumping. By Corollary 2.21, the left-invariant random walk driven by a weight w lumps exactly to G/H if, when started at the uniform distribution on G, the distribution conditioned on the sequence of observations of the value of the top card is uniform on the relevant left coset. By Corollary 1.10(ii), this holds if and only if $w(Hg)$ is constant for $g \in H(1, 2)H$, or equivalently, if and only if $w\big(H(1, 2)\big) = w\big(H(1, 3)\big) = w\big(H(1, 4)\big)$. Dualizing the previous remarks, this holds whenever each column in Figure 1 has equal weight, and in particular, whenever the non-identity part of w is uniformly distributed on any left coset of H. Taking one element of the left coset $(1, 2)H$ in each right coset of H, together with the identity, we obtain the *top-to-random* shuffle $\frac{1}{4}\mathrm{id} + \frac{1}{4}(1, 2) + \frac{1}{4}(1, 3, 2) +$

	$H(1,4)$	$H(1,2)$	$H(1,3)$
$(1,4)H$	$(1,4)$ $(1,4)(2,3)$	$(1,2,3,4)$ $(1,2,4)$	$(1,3,4)$ $(1,3,2,4)$
$(1,2)H$	$(1,4,3,2)$ $(1,4,2)$	$(1,2)$ $(1,2)(3,4)$	$(1,3,2)$ $(1,3,4,2)$
$(1,3)H$	$(1,4,3)$ $(1,4,2,3)$	$(1,2,3)$ $(1,2,4,3)$	$(1,3)$ $(1,3)(2,4)$

FIGURE 1. The double coset $H(1,2)H$ when $G = \mathrm{Sym}_4$ and $H = \mathrm{Sym}_{\{2,3,4\}}$. Rows are left cosets and columns are right cosets. For later use in §1.2.3, the double cosets TxT where $T = \mathrm{Sym}_{\{2,3\}}$ are coloured. Thus $T(1,3)T = (1,2)T \cup (1,3)T = \{(1,3),(1,2,3)\} \cup \{(1,3,2),(1,2)\}$ is dark blue and $T(1,2,3)T$ is light blue; in black and white any remaining ambiguity can be resolved by noting that since $(1,4) \in N_{\mathrm{Sym}_4}(T)$, we have $T(1,4) = (1,4)T$ and hence $H(1,4) = T(1,4) \cup T(1,4,3)T$ and $(1,4)H = (1,4)T \cup T(1,3,4)T$.

$\frac{1}{4}(1,4,3,2)$. See §10 for more on the time-reversal symmetry between the random-to-top and top-to-random shuffles.

1.2.3. Weak lumping: weights compatible with an idempotent. Consider the subgroup $T = \mathrm{Sym}_{\{2,3\}}$ of $H = \mathrm{Sym}_{\{2,3,4\}}$. The idempotent in $\mathbb{C}[\mathrm{Sym}_4]$ corresponding to the uniform distribution on T is $\eta_T = \frac{1}{2}\mathrm{id} + \frac{1}{2}(2,3)$. By Proposition 11.1, the left-invariant random walk driven by w lumps stably for $\mathbb{C}[G]\eta_T$, in the sense of Definition 1.3, if and only if the left-invariant random walk driven by a weight w lumps exactly on the left cosets G/T and $\frac{1}{|TgH|}w(TgH)$ is constant for $TgH \subseteq H(1,2)H$. In particular, if the non-identity part of w is supported on $H(1,2)H$, then it is necessary and sufficient that

- $w(Tg) = w\big(Tg(2,3)\big)$ for all $g \in H(1,2)H$;
- $w\big((1,4)H\big) = \frac{1}{2}w\big((1,2)H\big) + \frac{1}{2}w\big((1,3)H\big)$.

In particular, if $0 \le \lambda \le 1$ and

$$w = (1-\lambda)\mathrm{id} + \tfrac{\lambda}{3}\big((1,4)(2,3) + (1,4,3) + (1,4,2,3)\big) \qquad (1.3)$$

then both conditions hold. (For the first condition, note that $T(1,4) = \{(1,4)(2,3),(2,3)\} = T(1,4)(2,3)$ so the condition holds for $T(1,4)$, and since $(1,4,3)(2,3) = (1,4,2,3)$, so it holds for $T(1,4,3)$ and $T(1,4,2,3)$, the other relevant cosets.) But since w is neither constant on the rows nor on the columns in Figure 1, this weight does not lump strongly or exactly.

The probabilistic interpretation is as follows: ask a friend to take the deck of cards in its starting configuration, and, if a fair coin lands heads, swap the middle two cards. This gives an initial distribution corresponding to the idempotent $\eta_T \in \mathbb{C}[G]$. The deck is then shuffled repeatedly according to w, and after each shuffle, the top card is revealed. Since $\mathrm{MC}(\eta_T, w)$ lumps weakly, the sequence of values of the top card is a Markov chain on the set of cards $\{\mathrm{A}, \mathrm{K}, \mathrm{Q}, \mathrm{J}\}$. Thus the shuffle frustrates card counters: thanks to the Markov property, the card counter gets no benefit from memorizing the past history of the top card. Moreover the distribution, conditioned on the observation that the card starting in position k is on top, gives equal probability to g and $g(2,3)$ for each relevant g, namely those g such that $kg = 1$. In the special case when $\lambda = \frac{3}{4}$ and so w gives equal probability $\frac{1}{4}$ to each shuffle, the probabilities p_k that the card in position k moves to the top are

k	1	2	3	4
p_k	$\frac{1}{4}$	0	$\frac{1}{2}$	$\frac{1}{4}$

and it follows that, at every time, each card is equally likely to be at the top position. Therefore the sequence of top cards is not only Markov, but in fact independent and uniformly equidistributed. We remark that the entropy of w in this case is $-4 \times \frac{1}{4} \log_2 \frac{1}{4} = 2$; this is the least possible entropy of a shuffle that induces a weak lumping with these properties. The shuffle defined with $\lambda = \frac{1}{4}$, *with our chosen initial distribution*, is therefore not only frustrating to card counters, but also fair and efficient!

Finally, we show that while our chosen weight w, namely $(1-\lambda)\mathrm{id} + \frac{\lambda}{3}\big((1,4)(2,3) + (1,4,3) + (1,4,2,3)\big)$, lumps weakly, $\mathrm{MC}(\alpha, w)$ may fail to lump weakly to G/H if the initial distribution is outside $\mathbb{C}[\mathrm{Sym}_4]\eta_T$. Started deterministically at the identity, the pack reads QJAK top-to-bottom. We have

$$\mathbb{P}[X_3 = \mathrm{J} \mid X_2 = \mathrm{Q}, X_1 = \mathrm{Q}] = 0$$

since the event $\{X_2 = \mathrm{Q}, X_1 = \mathrm{Q}\}$ implies that each of the first two shuffles is the identity, and no shuffle in the support of w moves position 2 to position 1. On the other hand,

$$\mathbb{P}[X_3 = \mathrm{J} \mid X_2 = \mathrm{Q}] \neq 0,$$

since the sequence of shuffles

$$\mathrm{QJAK} \xmapsto{(1,4,3)} \mathrm{AJKQ} \xmapsto{(1,4)(2,3)} \mathrm{QKJA} \xmapsto{(1,4,3)} \mathrm{JKAQ}$$

is consistent with the event $\{X_2 = \mathrm{Q}\}$.

1.2.4. A variation and infinitely many weakly lumping shuffles. Take a deck of n cards, and shuffle as follows:

> Remove the bottom card, insert it under a random card chosen uniformly from the remaining deck, then move the top card to the bottom.

In the case $n = 4$, this is the shuffle driven by the weight w above, defined with $\lambda = 0$ so that the identity permutation has zero probability. By Proposition 11.3, this shuffle lumps weakly to left cosets of $\mathrm{Sym}_{\{2,\ldots,n\}}$ inside Sym_n, but, as seen in the special case $n = 4$, it does not lump strongly or exactly. Proposition 11.1 may be used to give many other such examples of infinite families of weakly lumping left-invariant random walks, parametrised by degree, that do not lump strongly or exactly.

1.2.5. Weak lumping: idempotents compatible with a weight. We have seen that the left-invariant random walk driven by the weight w defined in (1.3) weakly lumps for distributions in $\mathbb{C}[G]\eta_T$. Correspondingly, as expected from Theorem 1.2 in the case $e = \eta_T$, we have

$$\eta_T w(1-\eta_T) = 0,$$
$$(\eta_T - \eta_H)w\eta_H = 0.$$

Consider the weight $w' = \eta_T w = \big(\frac{1}{2}\,\mathrm{id} + \frac{1}{2}(2,3)\big)w$ corresponding to first swapping the middle two cards according to a coin-flip, and then shuffling according to w (defined with general λ). Since η_T is an idempotent and $\eta_H\eta_T = \eta_H$, the two displayed equations above hold replacing w with w'. Therefore w' satisfies the conditions of Theorem 1.2 for $e = \eta_T$ and, by this theorem, the weight w' also weakly lumps for distributions in $\mathbb{C}[G]\eta_H$. Using the definition $w = (1-\lambda)\frac{1}{4}\mathrm{id} + \lambda\big(\frac{1}{4}(1,4)(2,3)\big) + \frac{1}{4}(1,4,3) + \frac{1}{4}(1,4,2,3)\big)$ and the coset diagram in Figure 1, it is easy to see that

$$w' = \eta_T w = (1-\lambda)\big(\tfrac{1}{8}\,\mathrm{id} + \tfrac{1}{8}(2,3)\big) + \ \tfrac{\lambda}{8} \sum_{g \in H(1,2)} g$$

where the first two summands are $\frac{1-\lambda}{4}\eta_T$. It easily follows that w' satisfies the condition in Corollary 1.10(i) for strong lumping. This should be expected from the previous subsection, because w' includes the randomising effect of the choice of initial distribution in $\mathbb{C}[G]\eta_T$. This example also shows that weak lumping and stable lumping are still of interest, *even in the case when the left-invariant random walk lumps strongly.*

We use this example to illustrate the constructive methods of §6 in Examples 6.1 and 6.2.

1.3. **Why irreducible weights?** Figure 2 shows a pentagon whose vertices are labelled by the residue classes modulo 5 and whose front and back faces are marked F and B, respectively. Let σ be anticlockwise rotation by $2\pi/5$ and let τ be reflection through the vertical axis. These symmetries preserve the position of the pentagon in the plane and generate the dihedral group of order 10 with presentation

$$G = \langle \sigma, \tau : \sigma^5 = \tau^2 = (\sigma\tau)^2 = 1 \rangle.$$

Let $H = \langle \tau \rangle$. Observe that the left coset $\sigma^i H$ consists of the permutations σ^i and $\sigma^i\tau$ that move the vertex labelled $-i$ to the top position. Thus the left-invariant random walk on G lumps weakly on G/H if and only if the sequence of observations of the label at the top position forms a Markov chain. By Corollary 5.17, if the driving weight is irreducible, this holds if and only if the chain lumps exactly or strongly. The weakly lumping irreducible weights are therefore classified by Corollary 1.10. For example, the uniform weight on $\sigma H = \{\sigma, \sigma\tau\}$ lumps strongly; the uniform weight on $H\sigma = \{\sigma, \sigma^{-1}\tau\}$ lumps exactly, and the uniform weight on $\{\sigma\tau, \sigma^{-1}\tau\}$ lumps strongly *and* exactly; the latter is expected because the support is a conjugacy class of G; as we explain in the literature survey in §1.4 below, this follows from the main theorems of either [8] or [16].

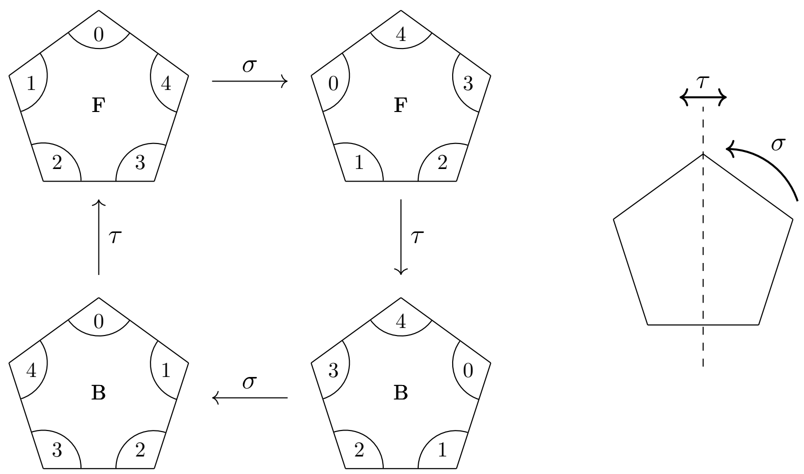


FIGURE 2. The pentagon in its fixed position on the plane showing the anti-clockwise rotation σ by $\pi/5$, the reflection τ and the sequence of symmetries $\sigma, \tau, \sigma, \tau$. The rotation σ acts on the vertex labels as the 5-cycle $(0, 4, 3, 2, 1)$ if the front face is uppermost and as the 5-cycle $(0, 1, 2, 3, 4)$ if the back face is uppermost.

Example 1.13. We take the weight $1+\sigma \in \mathbb{C}[G]$. If the initial distribution α is supported on $\langle\sigma\rangle$, respectively $\langle\sigma\rangle\tau$, then at every time, σ permutes the labels of the pentagon by the 5-cycle $(0, 4, 3, 2, 1)$, respectively $(0, 1, 2, 3, 4)$. (This is shown visually in Figure 2.) In the first case the induced process on the top label is the Markov chain on $\{0, 1, 2, 3, 4\}$ in which the steps $j \mapsto j-1 \bmod 5$ and $j \mapsto j$ are equally likely; the same holds in the second case replacing $j-1$ with $j+1$. Therefore $\mathrm{MC}(\alpha, w)$ lumps weakly to G/H. Conversely, suppose that $\alpha\big(\langle\sigma\rangle\big) > 0$ and $\alpha\big(\langle\sigma\rangle\tau\big) > 0$. The sequence of observations of the label at the top position is then of one of the two forms

$$\begin{gathered} i, \ldots, i, i+1, \ldots, i+1, i+2, \ldots, \\ i, \ldots, i, i-1, \ldots, i-1, i-2, \ldots \end{gathered}$$

where the first label seen other than i is $i-1$ if at time 0 face F was uppermost and $i+1$ if at time 0 face B was uppermost. Writing Y_t for the label at the top position at time t, and noting that it is possible that $Y_4 = 0$ and $Y_5 = 1$, we have $\mathbb{P}[Y_6 = 0 \mid Y_5 = 1, Y_4 = 0] = 0$ whereas $\mathbb{P}[Y_6 = 0 \mid Y_5 = 1] > 0$. Therefore $\mathrm{MC}(\alpha, 1+\sigma)$ lumps weakly if and only if α is supported on a single coset of $\langle\sigma\rangle$.

This example shows that, in the reducible case, the set of initial distributions α such that $\mathrm{MC}(\alpha, w)$ lumps weakly need not be convex. This rules out any routine generalization of our main result, Theorem 1.2, beyond the irreducible case. See also Example 2.1 for another illustration of the more complicated behaviour seen in the reducible case. We believe this case is of considerable interest and plan to study it further in a sequel to this paper.

1.4. **Earlier work.** For two excellent surveys of the vast literature on left-invariant random walks on finite groups we refer the reader to Diaconis [13] and Saloff-Coste [33]. In particular, by [33, Proposition 2.3] the left-invariant random walk on a finite group G driven by a weight w is irreducible if and only if the support of w generates G, and is aperiodic if and only if the support of w is not contained in a coset of a proper normal subgroup of G. As we saw in §1.2 the left-invariant random walk on a symmetric group models repeated shuffles of a deck of cards. Notable examples where representation theory has been used to analyse the mixing times of shuffles include the r-top-to-random shuffle studied in [7], [14], [18] and [29], and the riffle shuffle, studied in [1] and [4]. The recent book [15] is an excellent introduction to this area. The left-invariant random walks on general linear groups, or more broadly, finite groups of Lie type, have also been extensively studied: see [25, Theorems 1.8, 1.9] and [33, §9.4] for further references.

Weak lumping in general. We refer the reader to §2 for a comprehensive survey of weak lumping including a characterisation of weak lumping due to Gurvits and Ledoux [20] that is the basis of all our main results. Earlier introductory accounts include [24, §6.4] and [28, §2.4].

Weak lumping of the top-to-random shuffle. Applying the RSK correspondence (see for instance [19]) to a permutation $\sigma \in \mathrm{Sym}_n$ one obtains a pair of standard tableaux (S, T) of the same shape. For example, σ is a non-trivial top-to-random shuffle if and only if S and T have shape $(n-1, 1)$ and the unique entry in the second row of the insertion tableau S is 2; the entry in the second row of the recording tableau T is then the position of the top card after the shuffle. In [18, Theorem 3.1], Fulman proves that after t steps of the top-to-random shuffle, starting at the identity, the distribution of the RSK shape of the permutation agrees with the probability distribution after t steps of a random walk on partitions of n defined using Plancherel measure, started at (n). Since the RSK shape of the identity is (n), this might suggest the top-to-random shuffle lump strongly to partitions by taking RSK shapes, but by Proposition 7.2 in [7], this is *not* the case. However, in [28, Theorem 3.9], Pang showed that taking RSK shapes in a modified version of the shuffle *does* lump weakly on a subspace of distributions (as in Corollary 2.12) that are constant on sets of permutations having equal insertion tableau S. This leads to a conceptual proof via weak lumping of Fulman's result, which, as Fulman remarks on page 12 of [18] initially seemed 'quite mysterious'.

Strong lumping to right cosets. As we saw in §1.2, the left-invariant random walk lumps strongly on right cosets for any subgroup H of G. In [3, Appendix 1] (where left- and right- are swapped relative to this paper) the authors prove this fact and describes the transition matrix of the lumped walk. This fact is used in [28, Example 2.8] to show that the Markov chain on $\mathbb{F}_2^n$ driven by flipping a position chosen uniformly at random lumps strongly by deleting any single bit and in [28, Example 2.9] to give a shuffling example in which the lumped chain tracks the position of a chosen card. In [28, Example 2.11] the case $G = \mathrm{Sym}_n$ and $H = \mathrm{Sym}_r \times \mathrm{Sym}_{n-r}$ is used to show that the r-top-to-random shuffle lumps strongly by recording the *positions* of the r cards beginning at the top of the deck.

Strong lumping to left and double cosets. As we explain in §4, the *double cosets* of a finite group G for subgroups H and K are the subsets HxK for $x \in G$. In [16], Diaconis, Ram and Simper prove that if the weight w is conjugacy-invariant, that is $w(g) = w(t^{-1}gt)$ for all g, $t \in G$, then the left-invariant random walk lumps strongly to the double cosets $H\backslash G/K$ for any subgroups H and K. Theorem 1.2 in [16] characterises the stationary distribution of the lumped chain and gives a good bound on the mixing time. Later, in [8], Britnell and Wildon gave a necessary and sufficient condition for the left-invariant random walk to lump strongly to the double cosets $H\backslash G/K$. Their Corollary 1.2 implies that a left-invariant random walk lumps strongly to $H\backslash G/H$ if and only if it lumps strongly to the left cosets G/H; by Corollary 1.10(i) this holds if and only if $w(gH)$ is constant for left cosets gH in the same double coset.

Notable examples of strong lumping to left and double cosets. The r-random-to-top shuffle is the shuffle of a deck of n cards in which r cards, chosen uniformly at random, are moved to the top of the pack, while maintaining their relative order. As we saw in the special case $r = 1$ in §1.2, it lumps strongly to the left cosets of $\mathrm{Sym}_r \times \mathrm{Sym}_{n-r}$; the lumped Markov chain has states corresponding to the $\binom{n}{r}$ possible sets of labels of the top r cards in the deck, and in fact it always transitions to a uniform random state. In [8, §3.2] it is shown that a family of shuffles generalizing the r-random-to-top shuffle lumps strongly to the double cosets of $\mathrm{Sym}_r \times \mathrm{Sym}_{n-r}$ in Sym_n; the lumped random walks are reversible and have some remarkable spectral properties. The time reversal of the r-random-to-top shuffle is the r-top-to-random shuffle, in which the top r cards from the pack are moved to r positions in the pack chosen uniformly at random, again maintaining the relative order of the r moved cards and the relative order of the other $n-r$ cards. By Theorem 1.9, the r-top-to-random shuffle lumps exactly to the left cosets of $\mathrm{Sym}_r \times \mathrm{Sym}_{n-r}$ in Sym_n, and lumps exactly to the double cosets of $\mathrm{Sym}_r \times \mathrm{Sym}_{n-r}$ in Sym_n. These shuffles are notable because the weights are not conjugacy invariant (except in the trivial case $r = n$), and so the full power of the results from [8] and Theorem 1.9 is required.

We refer the reader to [16] for a wealth of further examples of strong lumping to double cosets of random walks driven by conjugacy invariant weights: particularly notable is Example 2.3, that the random walk on a finite group of Lie type driven by multiplication by such a weight lumps strongly to the double cosets of a Borel subgroup, and so induces a random walk on its Weyl group.

Weak lumping of other left-invariant random walks. Besides the result of Pang already mentioned, where the lumping is to partitions, we know of no substantial examples in the literature of weak lumpings of left-invariant random walks that are not also strong lumpings. In particular, the present paper is the first to study weak lumpings of the left-invariant random walk to left cosets; as our extended examples in §1.2 and §13.2 show, the theory we develop is broadly applicable. We hope this paper will lead to further interesting examples of weak lumping.

1.5. **Outline.** The outline of this paper is as follows.

- In §2 we present the necessary background from probability theory. In particular Theorem 2.6 and Corollary 2.12 give a short self-contained proof of a characterisation of weak lumping due to Gurvits and Ledoux [20].
- In §3 we give the necessary background from representation theory including a primer on induced representations and the Wedderburn decomposition.
- In §4 we collect basic results on the double coset decomposition of groups including a special case of Mackey's restriction formula.
- In §5 we use Theorem 2.6 to show that weak lumping of the right-invariant random walk is controlled by the behaviour of left ideals of $\mathbb{C}[G]$ of the form $\mathbb{C}[G]e$ where $e \in E^\bullet(H)$ and to prove Theorem 1.2, Corollary 1.5, Corollary 1.6 and Theorem 1.7.
- In §6 we give efficient computational algorithms for the tests in Corollary 1.6 and Theorem 1.7, illustrated by the running example in §1.2.
- In §7 we make a detailed study of the weak lumping algebras $\Theta(e)$ defined in (1.1) and prove Proposition 1.8.
- In §8 we prove Corollary 8.3, which shows that only idempotents with real coefficients need to be considered in Theorem 1.2 to obtain all weakly lumping weights.
- In §9 we prove Theorem 1.11.
- In §10 we state and prove Theorem 10.1, a new result relating duality and time reversal for weak lumping of general finite time-homogeneous Markov chains, and use it to prove Theorem 1.9 and Corollary 1.10.
- In §8 we prove Corollary 8.3, which shows that only idempotents with real coefficients need to be considered in Theorem 1.2 to obtain all weakly lumping weights.
- In §11 we prove Proposition 11.1; this generalizes part of the shuffling example in §1.2 and gives a rich supply of left ideals for which the left-invariant random walk lumps stably.
- In §12 we show how to interpret the conditions in Theorem 1.2 and our other main results working double coset by double coset.
- In §13 we finish by applying the results of the previous section to the case where H is abelian, obtaining a very sharp description of the weakly lumping weights in Proposition 13.2 and proving Corollary 1.12. We finish with an extended example in which $G = \mathrm{Sym}_4$ and $H = \langle(1, 2, 3, 4)\rangle$.

2. Background from Markov chain theory

In this paper we are concerned with *discrete time-homogeneous Markov chains* (DTHMCs) with finite state spaces. A DTHMC is specified by a state space A, a probability measure μ on A, and a stochastic matrix P whose rows and columns are indexed by the elements of A. We call μ the *initial distribution*, P the *transition matrix*, and we denote this DTHMC by $X = \mathrm{MC}(\mu, P)$. It is a random variable $X = (X_t)_{t\in\mathbb{N}_0}$ with the properties that

$$\mathbb{P}[X_0 = x] = \mu(x), \quad \text{for each } x \in A,$$

and for any $t \geq 1$ and any sequence $x_0, \ldots, x_t \in A$,

$$\mathbb{P}[X_0 = x_0, \ldots, X_t = x_t] = \mu(x_0)P(x_0, x_1) \ldots P(x_{t-1}, x_t).$$

It follows from this definition that for any $t \geq 0$, any $y \in A$, and any sequence $x_0, \ldots, x_t$ such that $\mathbb{P}[X_0 = x_0, \ldots, X_t = x_t] > 0$, we have

$$\mathbb{P}[X_{t+1} = y \mid X_0 = x_0, \ldots, X_t = x_t] = \mathbb{P}[X_{t+1} = y \mid X_t = x_t] = P(x_t, y).$$

In a predecessor to this paper, Britnell and Wildon [8], the term 'Markov chain' refers to a transition matrix, with the initial distribution left unspecified. This is how the term was used in the older probability literature, for example Kemeny and Snell [24]. In the recent probability literature, a 'Markov chain' more often means a stochastic process with a countable state space. This is the usage here.

We follow the usual convention of probabilists that transition matrices act on the right on row vectors. Thus the matrix entry $P(x, y)$ or $P_{x,y}$ stands for the conditional probability that our Markov chain steps next to y given that it is currently at x. For example, the left-invariant random walk driven by a weight w has transition matrix $P(x, y) = w(x^{-1}y)/w(G)$. If $w(G) = 1$ we say the weight is *normalized*. The statement that a probability distribution μ is *stationary* for P is written algebraically as $\mu P = \mu$. When μ is stationary for P and $X = \mathrm{MC}(\mu, P)$, we have for each $t \geq 0$ that $X_t \sim \mu$, meaning that X_t is distributed according to μ.

When $X = (X_t)_{t \geq 0}$ is a DTHMC with state space A and $f : A \to B$ is a function, we will write $f(X)$ to mean the discrete time stochastic process $(f(X_t))_{t \geq 0}$. The process $f(X)$ is *typically not a Markov chain*. This is neatly illustrated by the deterministic bottom-to-top shuffle of a deck of $n \geq 3$ cards that is initially uniformly distributed among the $n!$ possible orders. It is also possible for $f(X)$ to be a Markov chain without being time-homogeneous, as the following example shows.

Example 2.1 (Inhomogeneous Markovian image process)**.** Let $n > 2$ and let $G = \langle \sigma, \tau \mid \sigma^n, \tau^2, (\sigma\tau)^2 = 1 \rangle$ be the dihedral group of order $2n$, as seen in §1.3 in the case $n = 5$. Let $(X_t)_{t \geq 0}$ be the left-invariant random walk on G driven by the weight $w = \frac{1}{2}(\tau + \sigma\tau)$, with initial distribution α. This weight is irreducible, but the transition matrix of the walk is periodic. Let H be the order 2 subgroup $\langle \tau \rangle$. We get an inhomogeneous Markov process $(X_t H)_{t \geq 0}$ on G/H if the initial distribution α is either concentrated on $\langle \sigma \rangle$ or concentrated on $\langle \sigma \rangle \tau$. In either case, the transition matrices of $(X_t H)_{t \geq 0}$ form an alternating sequence. For any other initial distribution α, the process $(X_t H)_{t \geq 0}$ is not a Markov chain.

Definition 2.2 (Weak lumping with a given initial distribution)**.** When $X = \mathrm{MC}(\mu, P)$ and $f(X)$ is a DTHMC, we say that X *lumps weakly* under f, and we say that P *lumps weakly under f starting in distribution μ*.

Definition 2.3 (Weak lumpability)**.** A transition matrix P on a state space A is *weakly lumpable under f* if there exists an initial distribution μ on A such that $f(\mathrm{MC}(\mu, P))$ is a DTHMC.

If the map f is understood in the context, for example, when it is the quotient map from $A = G$ to $B = G/H$, we may also talk about weak lumping *to* B or *on* B instead of weak lumping under f. We avoid using the terms *lumping*, *lumpable*, and *lumpability* on their own, because in some of the literature (for instance [8]) this has been used to refer to the concept of strong lumping in the sense of Definition 2.18 below.

2.1. The Gurvits–Ledoux characterisation of weak lumping of time homogeneous Markov chains.

Gurvits and Ledoux [20] characterised weak lumping of DTHMCs with finite state spaces using linear algebra. They also considered higher order Markov chains and hidden Markov chains with probabilistic output functions. In their notation, transition matrices act on the left on column vectors. To establish our preferred notation and to keep our paper self-contained, we now give a rapid exposition of some results from [20] which we will use in our study of weak lumping of left-invariant random walks from G to G/H. We also introduce a new notion of *stable* weak lumping and develop some of its properties.

Let A and B be finite sets and $f : A \to B$ a surjective function. Then A is partitioned into the non-empty *lumps* $f^{-1}(b)$ for $b \in B$. The linear map $F : \mathbb{R}^A \to \mathbb{R}^B$ induced by f is defined on the canonical bases of $\mathbb{R}^A$ and $\mathbb{R}^B$ by $F(e_a) = e_{f(a)}$. Acting on the right on row vectors, F is represented by the matrix defined by $F_{a,b} = 1$ if $f(a) = b$ and 0 otherwise. For any vector subspace V of $\mathbb{R}^A$, we define

$$V^\circ = V \cap \ker F. \tag{2.1}$$

The space V° consists of those vectors in V such that the sum of the coordinates over the lump $f^{-1}(b)$ vanishes for each $b \in B$. We also define linear endomorphisms of $\mathbb{R}^A$ which act on the right on standard basis vectors (e_a for $a \in A$) by

$$e_a\Pi_b = \begin{cases} e_a & \text{if } f(a) = b, \\ 0 & \text{otherwise.} \end{cases}$$

Thus Π_b is the projection onto the direct summand indexed by b in the direct sum

$$\mathbb{R}^A = \bigoplus_{b\in B} \mathbb{R}^{f^{-1}(b)}.$$

Since we are using row vectors, the matrices P, F and Π_b act on the right. When V is a linear subspace of $\mathbb{R}^A$ and M is a linear map with domain $\mathbb{R}^A$, we write

$$VM = \{vM : v \in V\}.$$

We also use the above notations adapted to complex vector spaces.

Definition 2.4. If V is a real linear subspace of $\mathbb{R}^A$ or a complex linear subspace of $\mathbb{C}^A$ then we call V a *Gurvits–Ledoux space* if it satisfies $VP \subseteq V$ and $V\Pi_b \subseteq V$ for every $b \in B$.

Every Gurvits–Ledoux space V satisfies $V = \bigoplus_{b\in B} V\Pi_b$.

Definition 2.5. For $P \in \mathrm{Mat}(\mathbb{R}^A)$ a stochastic matrix and $\alpha \in \mathbb{R}^A$ a probability distribution, let $V(f, P, \alpha)$ be the minimal vector space $V \subseteq \mathbb{R}^A$ such that

(a) $\alpha \in V$,
(b) $VP \subseteq V$,
(c) $V\Pi_b \subseteq V$ for all $b \in B$.

This definition makes sense because each of conditions (a)–(c) is closed under intersection and $V = \mathbb{R}^A$ satisfies (a)–(c). Note that $V(f,P,\alpha)$ is the minimal real Gurvits–Ledoux space containing α. It is straightforward to check that the minimal complex Gurvits–Ledoux space containing α is $\mathbb{C} \otimes_{\mathbb{R}} V(f,P,\alpha)$.

Readers who wish to compare our exposition with that of Gurvits and Ledoux [20] will find that they take $A = \{1,\ldots,N\}$ and $B = \{1,\ldots,M\}$, their map φ is our f, and the space we have called $V(f,P,\alpha)$ is called $\mathcal{CS}(\alpha,\Pi_\cdot,P)$ there. We shall not comment further on the translation between [20] and our notation, but simply state their results in our notation.

We have $\ker F = \bigoplus_{b\in B} (\ker F)\Pi_b$, so

$$V(f,P,\alpha)^\circ = \bigoplus_{b\in B} \big((V(f,P,\alpha)\Pi_b) \cap \ker F\big) = \bigoplus_{b\in B} V(f,P,\alpha)^\circ\, \Pi_b. \qquad (2.2)$$

Considering $X = \mathrm{MC}(\alpha,P)$, we say that a finite sequence $(b_0,\ldots,b_t)$ of elements of B is an *α-possible sequence* if $\mathbb{P}[f(X_0) = b_0,\ldots,f(X_t) = b_t] > 0$. For any such sequence, let $C(\alpha; b_0,\ldots,b_t)$ be the conditional distribution of X_t given $f(X_0) = b_0,\ldots,f(X_t) = b_t$, thought of as a row vector in $\mathbb{R}^A$. The vector space $V(f,P,\alpha)$ is the minimal linear subspace of $\mathbb{R}^A$ that contains $C(\alpha; b_0,\ldots,b_t)$ for every α-possible sequence $(b_0,\ldots,b_t)$.

Notice that $f(X)$ is a time-homogeneous Markov chain if and only if the conditional distribution of $f(X_{t+1})$ given $f(X_0) = b_0,\ldots,f(X_t) = b_t$ depends only on the value b_t, and is given by the same function for all times t. In other words, $f(X)$ is a time-homogeneous Markov chain if and only if

$$C(\alpha; b_0,\ldots,b_t)PF = C(\alpha; b'_0,\ldots,b'_{t'})PF$$

whenever $(b_0,\ldots,b_t)$ and $(b'_0,\ldots,b'_{t'})$ are α-possible sequences and $b_t = b'_{t'}$. For each $b \in B$, let

$$S(\alpha,b) = \big\{C(\alpha; b_0,\ldots,b_t) : t \geq 0,\ b_t = b,\ (b_0,\ldots,b_t)\ \alpha\text{-possible}\big\}.$$

Observe that

$$V(f,P,\alpha)\Pi_b = \big\langle S(\alpha,b)\big\rangle.$$

where, as ever, angled brackets denote the linear span of a set of vectors. Hence

$$V(f,P,\alpha)^\circ\, \Pi_b = V(f,P,\alpha)\Pi_b \cap \ker F = \big\langle \mu - \nu : \mu,\nu \in S(\alpha,b)\big\rangle.$$

We see that the following are equivalent:

- $f(X)$ is a time-homogeneous Markov chain,
- for all $b \in B$, and for all $\mu,\nu \in S(\alpha,b)$, we have $(\mu-\nu)PF = 0$,
- for all $b \in B$, we have $V(f,P,\alpha)^\circ\, \Pi_b PF = \{0\}$, and
- $V(f,P,\alpha)^\circ PF = 0$.

The third and fourth conditions above are equivalent by (2.2). Finally,

$$V(f,P,\alpha)^\circ P \subseteq V(f,P,\alpha)P \subseteq V(f,P,\alpha),$$

so we have

$$V(f,P,\alpha)^\circ PF = 0 \iff V(f,P,\alpha)^\circ P \subseteq V(f,P,\alpha)^\circ.$$

We have proved the following result of Gurvits and Ledoux.

Theorem 2.6 (Gurvits and Ledoux [20, Corollary 9])**.** *The Markov chain* $\mathrm{MC}(\alpha,P)$ *lumps weakly under* f *if and only if* $V(f,P,\alpha)^\circ P \subseteq V(f,P,\alpha)^\circ$.

Now consider some non-empty set $\mathcal{A} \subset \mathbb{R}^A$ of probability vectors. We ask whether the image process $f(\mathrm{MC}(\alpha,P))$ is a DTHMC for every $\alpha \in \mathcal{A}$, with a transition matrix that does not depend on the choice of α from $\mathcal{A}$.

Definition 2.7. For $P \in \mathrm{Mat}(\mathbb{R}^A)$ a stochastic matrix and $\mathcal{A} \subset \mathbb{R}^A$ a non-empty set of probability vectors, $V(f,P,\mathcal{A}) = \sum_{\alpha\in\mathcal{A}} V(f,P,\alpha)$.

It is straightforward to see that $V(f,P,\mathcal{A})$ is the minimal Gurvits–Ledoux subspace of $\mathbb{R}^A$ that contains every element of $\mathcal{A}$.

Theorem 2.8 (Gurvits and Ledoux ([20, Corollary 11]))**.** *For a non-empty set* $\mathcal{A}$ *of probability vectors in* $\mathbb{R}^A$*, the following are equivalent:*

(a) *there exists a stochastic matrix* Q *from* B *to* B *such that for every* $\alpha \in \mathcal{A}$*,* $f(\mathrm{MC}(\alpha,P))$ *is a time-homogeneous Markov chain with transition matrix* Q*,*
(b) $V(f,P,\mathcal{A})^\circ P \subseteq V(f,P,\mathcal{A})^\circ$*,*
(c) $V(f,P,\mathcal{A})^\circ PF = 0$*.*

Proof. Since $V(f,P,\mathcal{A})P \subseteq V(f,P,\mathcal{A})$, (b) and (c) are equivalent. Condition (a) implies that $V(f,P,\alpha) \subseteq \ker(PF - FQ)$ for each $\alpha \in \mathcal{A}$, so $V(f,P,\mathcal{A}) \subseteq \ker(PF - FQ)$. Since $V(f,P,\mathcal{A})^\circ F = 0$ by definition, we deduce $V(f,P,\mathcal{A})^\circ PF = 0$. Thus (a) implies ($c$). Now assume (c) and let $\alpha \in \mathcal{A}$. We have $V(f,P,\alpha)^\circ \subseteq V(f,P,\mathcal{A})^\circ$ so $V(f,P,\alpha)^\circ PF = 0$ and $V(f,P,\alpha)^\circ P \subseteq V(f,P,\alpha)^\circ$ and, by Theorem 2.6, $\mathrm{MC}(\alpha,P)$ lumps weakly under f. Let Q^α be the transition matrix of $f(\mathrm{MC}(\alpha,P))$, which is defined only on the set

$$\{f(x) : (\alpha P^m)_x > 0 \text{ for some } m \geq 0\}.$$

To complete the proof that (c) implies (a) we must show that the y-rows of Q^α and Q^β agree, for any $\alpha,\beta \in \mathcal{A}$ and any $y \in B$ such that row y is defined for both Q^α and Q^β. Choose $m \geq 0$ such that $v = \alpha P^m \Pi_y \neq 0$, and $n \geq 0$ such that $w = \beta P^n \Pi_y \neq 0$. Then v and w are both supported on the lump $f^{-1}(y)$, and $w(f^{-1}(y))v - v(f^{-1}(y))w \in V(f,P,\mathcal{A})^\circ$, so

$$\begin{aligned} 0 &= \big(w\big(f^{-1}(y)\big)v - v\big(f^{-1}(y)\big)w\big)PF \\ &= w\big(f^{-1}(y)\big)vFQ^\alpha - v\big(f^{-1}(y)\big)wFQ^\beta \\ &= w\big(f^{-1}(y)\big)v\big(f^{-1}(y)\big)e_y(Q^\alpha - Q^\beta). \end{aligned}$$

Hence the y-rows of Q^α and Q^β coincide. It follows that (c) implies (a). □

The matrix Q in the above theorem may not be uniquely determined, because there may be some element $b \in B$ such that for any $\alpha \in \mathcal{A}$ the Markov chain $\mathrm{MC}(\alpha,P)$ never visits $f^{-1}(b)$.

We shall also make use of the following consequences of Theorem 2.8.

Corollary 2.9 (Algorithm to test for weak lumping of $\mathrm{MC}(\alpha, P)$ under f)**.** *Begin by setting* $V_0 = \langle \alpha\Pi_b : b \in B\rangle$*. Then, inductively for* $t = 0, 1, 2, \ldots$*, define*

$$V_{t+1} = V_t + \sum_{b\in B} V_t P \Pi_b.$$

Let $k = \inf\{t \geq 0 : V_{t+1} = V_t\}$*. Then* $k \leq |A| - 1$*,* $V_n = V_k$ *for all* $t > k$*, and* $V_k = V(f, P, \alpha)$*. Hence* $\mathrm{MC}(\alpha, P)$ *lumps weakly under* f *if and only if* $V_k^\circ PF = \{0\}$*.*

Proof. The assertion that $k \leq |A| - 1$ follows from the fact that $\dim V_0 \geq 1$, $V_{t+1} \supseteq V_t$ and therefore $\dim V_{t+1} \geq \dim V_t$, for each $t \geq 0$, and $V_t \subseteq \mathbb{R}^A$ so $\dim V_t \leq |A|$ for all t. So

$$k = \inf\{t \geq 0 : \dim V_{t+1} = \dim V_t\} \leq |A| - \dim V_0 \leq |A| - 1.$$

From the inductive definition of V_{t+1} in terms of V_t it is clear that $V_{k+1} = V_k$ implies $V_t = V_k$ for all $t > k$. Each V_t is the direct sum of its Π_b-projections, and each V_t contains α. Moreover,

$$V_k P \subseteq \sum V_k P \Pi_b \subseteq V_{k+1} = V_k$$

so $V = V_k$ is a vector space satisfying conditions (a)–(c) of Definition 2.5. To show that V_k is the minimal such space, check by induction on t that any such V must contain V_t for every $t \geq 0$, and in particular $V \supseteq V_k$. □

Definition 2.10 (Stable lumping)**.** Let $P \in \mathrm{Mat}(\mathbb{R}^A)$ be a stochastic matrix and let V be any real vector subspace of $\mathbb{R}^A$ or complex vector subspace of $\mathbb{C}^A$ such that

(a) V contains at least one probability vector.
(b) $VP \subseteq V$,
(c) $V\Pi_b \subseteq V$ for all $b \in B$,
(d) $V^\circ PF = 0$ or equivalently $V^\circ P \subseteq V^\circ$, where $V^\circ := V \cap \ker F$.

Then we say that P *lumps weakly under* f *with stable space* V, or more briefly that P *lumps stably for* V.

If $\mathrm{MC}(\alpha, P)$ lumps weakly under f then P lumps stably for the real vector space $V(f, P, \alpha)$. The same matrix P may also lump stably for other spaces $V \subseteq \mathbb{R}^A$ such that $V(f, P, \alpha) \subseteq V$. If P lumps stably for a complex vector space $V \subseteq \mathbb{C}^A$ then P also lumps stably for the real vector space $\mathbb{R}^A \cap V$. On the other hand, if P lumps stably for a real vector space $V \subseteq \mathbb{R}^A$ then P also lumps stably for the complexification $V \otimes_{\mathbb{R}} \mathbb{C}$. Our reason for considering complex vector spaces in this paper is that, as remarked in the introduction, we wish to make use of results in representation theory that hold for the group algebra $\mathbb{C}[G]$, but do not hold for $\mathbb{R}[G]$, because $\mathbb{R}$ is not algebraically closed.

Remark 2.11. We show below (see Proposition 5.12) that Definition 2.10 is compatible with Definition 1.3 in the case where $P = P_w$ is the transition matrix of a left-invariant random walk on a finite group G driven by a weight w, f is the natural map $G \to G/H$, and the space $V \subseteq \mathbb{C}^G$ is a left ideal $\mathbb{C}[G]e$ of $\mathbb{C}[G]$, where $e \in E^\bullet(H)$. That is, w lumps stably for $\mathbb{C}[G]e$ in the sense of Definition 1.3 if and only if P_w lumps stably for V in the sense of Definition 2.10.

The next result is a corollary of Theorem 2.6; a partial converse to it is Lemma 2.17.

Corollary 2.12 (Stable spaces as certificates of weak lumping)**.** *Suppose P lumps weakly under f with (real or complex) stable space V, and let α be any probability vector in V. Then $X = \mathrm{MC}(\alpha, P)$ lumps weakly under f. Moreover, for each $t \geq 0$ the conditional distribution of X_t given $f(X_0), \dots, f(X_t)$ always lies in V.*

Proof. Conditions (b) and (c) in Definition 2.10 imply that $V(f,P,\alpha) \subseteq V$, and (d) implies that $(V \cap \ker F)P \subseteq \ker F$, so

$$V^\circ(f,P,\alpha)P = \big(V(f,P,\alpha) \cap \ker F\big)P \subseteq (V \cap \ker F)P \subseteq \ker F.$$

We also have from the definition of $V(f,P,\alpha)$ that $V(f,P,\alpha)P \subseteq V(f,P,\alpha)$, so $V^\circ(f,P,\alpha)P \subseteq V^\circ(f,P,\alpha)$, which by Theorem 2.6 implies that $\mathrm{MC}(\alpha,P)$ lumps weakly under f.

For any α-possible sequence $(b_0,\dots,b_t)$, the conditional distribution of X_t given $f(X_0) = b_0, \dots, f(X_t) = b_t$ is the normalisation of the vector v, where $v = \alpha\Pi_{b_0}$ if $t = 0$, and if $t = 1$ then $v = \alpha\Pi_{b_0}(P\Pi_{b_1})\dots(P\Pi_{b_t})$. By induction using (b) and (c), we have $v \in V$. □

2.2. **The irreducible case.**

Lemma 2.13. *If $P \in \mathrm{Mat}(\mathbb{R}^A)$ is an irreducible stochastic matrix and μ is its unique stationary distribution then $\mu \in V(f,P,\alpha)$ for every probability distribution α on A. Moreover, if P lumps stably for V, then $\mu \in V$.*

Proof. By the ergodic theorem for irreducible Markov chains (see for instance [24, Theorem 5.1.2(b)]), we have $\frac{1}{n}\sum_{i=0}^{n-1} \alpha P^i \to \mu$ as $n \to \infty$. The terms in this convergent sequence belong to $V(f,P,\alpha)$ by properties (a) and (b) in Definition 2.5. Since $V(f,P,\alpha)$ is a finite-dimensional vector space, it is topologically closed, hence $\mu \in V(f,P,\alpha)$. If P lumps stably for V then we may pick a probability vector $\alpha \in V$ and then $V(f,P,\alpha) \subseteq V$ so $\mu \in V$. □

Corollary 2.14 (Weak lumpability of irreducible transition matrices)**.** *If $P \in \mathrm{Mat}(\mathbb{R}^A)$ is an irreducible stochastic matrix and μ is its unique stationary distribution then P is weakly lumpable under $f : A \to B$ if and only if $\mathrm{MC}(\mu,P)$ lumps weakly under f.*

Proof. If $\mathrm{MC}(\mu,P)$ lumps weakly under f then P is weakly lumpable under f, by the definition of weak lumpability. For the converse, suppose that $\mathrm{MC}(\alpha,P)$ lumps weakly under f. Then $\mu \in V(f,P,\alpha)$ by Lemma 2.13. It follows that $V(f,P,\mu) \subseteq V(f,P,\alpha)$ and hence $V^\circ(f,P,\mu) \subseteq V^\circ(f,P,\alpha)$ and therefore $V^\circ(f,P,\mu)P \subseteq \ker F$. Since also

$$V^\circ(f,P,\mu)P \subseteq V(f,P,\mu)P \subseteq V(f,P,\mu),$$

we find

$$V^\circ(f,P,\mu)P \subseteq V(f,P,\mu) \cap \ker F = V^\circ(f,P,\mu)$$

which by Theorem 2.6 implies that $\mathrm{MC}(\mu,P)$ lumps weakly under f. □

We remark that Corollary 2.14 was known long before the work of Gurvits and Ledoux: see for example [24, Theorem 4.6.3]. We believe that the following observation is new.

Lemma 2.15 (Lattice of stable spaces for a weakly lumpable irreducible transition matrix)**.** *Let $P \in \mathrm{Mat}(\mathbb{R}^A)$ be an irreducible stochastic matrix that is weakly lumpable under $f : A \to B$, with unique stationary distribution μ. The set of real subspaces of $\mathbb{R}^A$ for which P lumps stably is a lattice under intersection and sum, with bottom element $V(f, P, \mu)$. Likewise the set of complex subspaces of $\mathbb{C}^A$ for which P lumps stably is a lattice with bottom element $V(f, P, \mu) \otimes_{\mathbb{R}} \mathbb{C}$.*

Proof. The proof of Corollary 2.14 shows that when P is irreducible, every V for which P lumps stably must contain μ and hence all of $V(f, P, \mu)$, and if V is complex, then $V(f, P, \mu) \otimes_{\mathbb{R}} \mathbb{C} \subseteq V$.

Next, consider two spaces U and V (both real or both complex) such that P lumps stably for U and also for V. Let $V = U \cap V$. Then $\mu \in U$ and $\mu \in V$ so $\mu \in U \cap V$. We have $VP \subseteq UP \subseteq U$ and likewise $VP \subseteq V$ so $VP \subseteq V$. For any $b \in B$, $V\Pi_b \subseteq U\Pi_b \subseteq U$ and likewise $V\Pi_b \subseteq V$ so $V\Pi_b \subseteq V$. Finally, $V^\circ = U \cap V \cap \ker F = U^\circ \cap V^\circ$ so $V^\circ PF \subseteq U^\circ PF = 0$. We have shown that P lumps stably for V.

Now let $W = U + V$. Then $\mu \in W$ and $WP = UP + VP \subseteq U + V = W$. We have

$$W = U + V = \bigoplus_{b \in B} U\Pi_b + \bigoplus_{b \in B} V\Pi_b = \bigoplus_{b \in B} (U\Pi_b + V\Pi_b)$$

so

$$W\Pi_b = U\Pi_b + V\Pi_b \subseteq U + V = W$$

hence $W = \bigoplus_{b \in B} W\Pi_b$. Since P is irreducible and f is surjective, for each $b \in B$ we have $\mu(f^{-1}(b)) > 0$. By Lemma 2.13 we have $\mu \in U$ and $\mu \in V$, and so $U\Pi_b = \langle \mu\Pi_b \rangle \oplus U^\circ\Pi_b$, for each $b \in B$, and likewise for V. Hence

$$W\Pi_b = (U^\circ\Pi_b + V^\circ\Pi_b) \oplus \langle \mu\Pi_b \rangle,$$

and therefore $W^\circ = U^\circ + V^\circ$. Finally, $W^\circ PF = U^\circ PF + V^\circ PF = 0 + 0$. We have shown that P lumps stably for W. □

It follows that for any irreducible P that is weakly lumpable under f, there exists a unique maximal space $V_{\max}(f, P)$ for which P lumps stably. It contains $V(f, P, \alpha)$ as a subspace for every initial distribution α such that $\mathrm{MC}(\alpha, P)$ lumps weakly under f.

Corollary 2.16 (Initial distributions compatible with an irreducible weight)**.** *Let $P \in \mathrm{Mat}(\mathbb{R}^A)$ be an irreducible stochastic matrix with unique stationary distribution μ. Suppose* $\mathrm{MC}(\mu, P)$ *lumps weakly under the surjective map $f : A \to B$. Let $\mathcal{I}$ be the set of probability distributions α such that* $\mathrm{MC}(\alpha, P)$ *lumps weakly under f. Then $\mathcal{I}$ is equal to the convex polytope $\Delta \cap V_{\max}(f, P)$, where Δ is the simplex of probability vectors in $\mathbb{R}^A$. Moreover, there exists a unique stochastic matrix $Q \in \mathrm{Mat}(\mathbb{R}^B)$ that serves as a transition matrix for $f(\mathrm{MC}(\alpha, P))$ for every $\alpha \in I$. This Q is given by*

$$Q(i, j) = \frac{1}{\mu(f^{-1}(i))} \sum_{x \in f^{-1}(i)} \sum_{y \in f^{-1}(j)} \mu(x) P(x, y). \tag{2.3}$$

Proof. The characterisation of $\mathcal{I}$ is a straightforward corollary of Lemma 2.15 and Corollary 2.12. The existence of a single transition matrix Q from B to B which serves as a transition matrix for $f(\mathrm{MC}(\alpha, P))$ for every $\alpha \in \mathcal{I}$

follows from Theorem 2.8, by taking $\mathcal{A} = \mathcal{I}$ and noting that then $\mathcal{I} \subset V(f, P, \mathcal{I}) \subseteq V_{\max}(f, P)$ so condition (c) of Theorem 2.8 is satisfied. Because P is irreducible and f is surjective, there is in fact a unique transition matrix for $f(\mathrm{MC}(\alpha, P))$, for each $\alpha \in \mathcal{I}$. The expression (2.3) for Q in terms of P and μ is obtained by calculating

$$\mathbb{P}[f(X_1) = j \mid f(X_0) = i] = \frac{\mathbb{P}[f(X_0) = i \text{ and } f(X_1) = j]}{\mathbb{P}[X_0 = i]}.$$

Note that by the irreducibility of P and the surjectivity of f, the denominator $\mathbb{P}[X_0 = i] = \mu(f^{-1}(i))$ is non-zero for each $i \in B$. □

Let us emphasise that in the irreducible case the transition matrix for the lumped process $f(\mathrm{MC}(\alpha, P))$ is independent of the initial distribution α. In the reducible case this may fail, as we saw in Example 1.13. It can even fail in the case where $\mathrm{MC}(\alpha, P)$ lumps weakly under f for every initial distribution α. An example of this is given by $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 4\}$, $f(1) = f(2) = 1$, $f(3) = 3$, $f(4) = 4$ and

$$P = \begin{pmatrix} 0 & 0 & 2/3 & 1/3 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Returning to the setting of Corollary 2.16, it follows that $V_{\max}(f, P)$ is the maximal vector subspace $V \subseteq \mathbb{R}^A$ with the properties

(1) $VP \subseteq V$,
(2) $V\Pi_b \subseteq V$ for each $b \in B$, and
(3) $V \subseteq \ker(PF - FQ)$.

The space $V_{\max}(f, P)$ may therefore be computed by the following linear algebra algorithm. To begin, set

$$V = \bigoplus_{b \in B} \left\{ v \in \mathbb{R}^{f^{-1}(b)} : v(PF - FQ) = 0 \right\}.$$

Note that this V satisfies condition (3) above. While $VP \not\subseteq V$, replace V by

$$V \cap \bigoplus_{b \in B} \left\{ v \in \mathbb{R}^{f^{-1}(b)} : vP \in V \right\}.$$

The dimension of V cannot increase under this operation, and if it does not decrease then

$$V = \bigoplus_{b \in B} \left\{ v \in \mathbb{R}^{f^{-1}(b)} : vP \in V \right\}$$

and hence V satisfies conditions (1)–(3) above, i.e. P lumps stably for V. If W is any space for which P lumps stably, then throughout the algorithm we have $W \subseteq V$. Therefore the final V is equal to $V_{\max}$. By the assumption that $\mathrm{MC}(\mu, P)$ lumps weakly under f, we have $\langle \Pi_b \mu : b \in B \rangle \subseteq V_{\max}$, hence $\dim V_{\max} \geq |B|$. Therefore the algorithm terminates after no more than $|A| - |B|$ replacement steps.

We end with a new result that we shall use in the proof of Proposition 8.5.

Lemma 2.17 (Probabilistic characterisation of stable lumping)**.** *Let $P \in \mathrm{Mat}\left(\mathbb{R}^A\right)$ be an irreducible stochastic matrix, let $f : A \to B$ be a surjection, and let $V \subseteq \mathbb{R}^A$ be a linear subspace containing at least one probability vector. Suppose that for every probability vector $\alpha \in V$ the Markov chain $X = \mathrm{MC}(\alpha, P)$ satisfies both*

(i) *X lumps weakly under f, and*

(ii) *for $t \geq 0$, the conditional distribution of X_t given $f(X_0), \ldots, f(X_t)$ always lies in V.*

Then P lumps weakly under f with stable space V.

Proof. By hypothesis, V contains at least one probability vector, say α, so condition (a) of Definition 2.10 is satisfied. We may apply condition (ii) for $X = \mathrm{MC}(\alpha, P)$ and average over the distribution of $(f(X_0), \ldots, f(X_t))$, to see that the distribution of X_t lies in V for every t. By considering the mean of the distributions of $X_0, \ldots, X_t$ and taking the limit as $t \to \infty$, and using that V is closed, we find that the unique stationary distribution μ for P also lies in V. Now take an arbitrary $\beta \in V$. For sufficiently small ε we have a strictly positive probability vector $\alpha = (\mu + \varepsilon\beta)/(1 + \varepsilon \sum_{a \in A} \beta_a)$, to which we may apply condition (ii). Looking at the distribution of X_1 where $X = \mathrm{MC}(\alpha, P)$, we find that $\alpha P \in V$. Since $\mu P = P$, we deduce $\beta P \in V$, verifying condition (b). Considering the conditional distribution of X_0 given $f(X_0) = b$ (which is well-defined for each $b \in B$ because α has strictly positive coordinates) we deduce that $(\mu + \varepsilon\beta)\Pi_b \in V$ for each $b \in B$. This holds for all sufficiently small ε, so $\beta\Pi_b \in V$, verifying condition (c). Finally, let $\gamma \in V^\circ$, so $\gamma F = 0$ and in particular $\sum_{a \in A} \gamma_a = 0$. Then $\alpha^+ = \mu + \varepsilon\gamma$ and $\alpha^- = \mu - \varepsilon\gamma$ are probability vectors in V for sufficiently small ε, so by condition (i) and Corollary 2.16 we have $\alpha^+, \alpha^- \in V_{\max}(f, P)$. We have $\alpha^+ F = \alpha^- F$ by construction, so $\alpha^+ - \alpha^- = 2\varepsilon\gamma \in V_{\max}(f, P)^\circ$ and hence $\gamma P F = 0$, verifying condition (d). □

2.2.1. *Earlier work of Rubino and Sericola.* The set $\mathcal{I}$ described in Corollary 2.16 was first characterised by Rubino and Sericola in [31, 32] by an algorithm that computes the set of extreme vertices of $\mathcal{I}$. As far as we know, this algorithm may have exponential worst-case complexity because the number of extreme points of a polytope can be exponentially large in the dimension and the number of faces of the polyhedron. Indeed, the dual version of McMullen's upper bound theorem says that a convex polyhedron of dimension d defined by N linear inequalities may have at most

$$\binom{N - \lceil d/2 \rceil}{\lfloor d/2 \rfloor} + \binom{N - \lfloor d/2 \rfloor - 1}{\lceil d/2 \rceil}$$

vertices, and this is sharp; see [27, §5.4 and §5.5]. In addition, Rubino and Sericola gave an explicit set of $\left(|B| + |B|^2 + \cdots + |B|^{|A|}\right)$ equations in the entries of α and P, linear in α and polynomial of degree at most $|A|$ in P, such that $\mathrm{MC}(\alpha, P)$ lumps weakly under f if and only if all of these equations are satisfied by (α, P). The set of such pairs (α, P) is therefore a semi-algebraic set. More precisely, it is the intersection of an affine algebraic variety with a product of simplices.

2.3. **Strong lumping.** Strong lumping is a special case of weak lumping, defined by a simple algebraic condition on the transition matrix.

Definition 2.18. A transition matrix P on A *lumps strongly* under a map $f : A \to B$ if whenever $f(a) = f(a')$, for every $b \in B$ we have

$$\sum_{x \in f^{-1}(b)} P(a, x) = \sum_{x \in f^{-1}(b)} P(a', x).$$

This property is also known as *Dynkin's condition.* If P lumps strongly under f, then for every probability distribution α on A, the time-homogeneous Markov chain $\mathrm{MC}(\alpha, P)$ lumps weakly under f. In fact, when Dynkin's condition holds then P lumps stably for $V = \mathbb{R}^A$. Indeed, conditions (a)–(c) of Definition 2.10 hold trivially, and for condition (d) we must check that $(\ker F)PF = 0$. Since $\ker F$ is spanned by vectors of the form $e_a - e_{a'}$ where $f(a) = f(a')$, this is equivalent to Dynkin's condition. Now apply Corollary 2.12.

For the case of a left-invariant random walk on a finite group G, a simple necessary and sufficient condition for strong lumping to G/H was given by Britnell and Wildon [8]; see Corollary 1.10.

2.4. **Exact lumping.** Exact lumping is another special case of weak lumping. It is the opposite extreme: in strong lumping the stable space V certifying weak lumping is as large as possible, satisfying $V = \mathbb{R}^A$, while in exact lumping the stable space is as small as possible, satisfying $V^\circ = 0$, where V° is as defined in (2.1).

Definition 2.19. Let $\alpha \in \mathbb{R}^A$ be a probability distribution and P a transition matrix from A to A. Then we say that $\mathrm{MC}(\alpha, P)$ *lumps exactly* under $f : A \to B$ if $V(f, P, \alpha)^\circ = 0$. In the case where P is irreducible, with unique stationary distribution μ, we say that P *lumps exactly* under f when $\mathrm{MC}(\mu, P)$ lumps exactly under f.

Exact lumping implies weak lumping, by Theorem 2.6, because

$$V(f, P, \alpha)^\circ = 0 \implies V(f, P, \alpha)^\circ\, P \subseteq V(f, P, \alpha)^\circ.$$

Lemma 2.20. *Consider the stationary distribution* $\mu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \alpha P^i$. *The following are equivalent:*

(a) $\mathrm{MC}(\alpha, P)$ *lumps exactly under* f,
(b) $\dim(V(f, P, \alpha)\Pi_b) \le 1$ *for all* $b \in B$,
(c) $V(f, P, \alpha) = \bigoplus_{b \in B} \langle \mu \Pi_b \rangle$.

Proof. Suppose for a contradiction that (a) holds but (b) does not, that is to say $V(f, P, \alpha)^\circ = 0$ and there exist linearly independent vectors v and v' in $V(f, P, \alpha)\Pi_b$. Then $v\Pi_b = ce_b$ and $v'\Pi_b = c'e_b$ for some $c, c' \in \mathbb{R}$. So either $c' = c = 0$, in which case v is a non-zero vector in $V(f, P, \alpha)$, or $c'v - cv'$ is a non-zero vector in $V(f, P, \alpha)$, a contradiction. Thus (a) implies (b).

To show that (b) implies (c), note $\mu \in V(f, P, \alpha)$, so $V(f, P, \alpha) \supseteq \bigoplus_{b \in B} \langle \mu \Pi_b \rangle$. Now for each lump $f^{-1}(b)$ that is accessible from some point in the support of α, we have a one-dimensional space $V(f, P, \alpha)\Pi_b$, which must be $\langle \mu \Pi_b \rangle$, and for each inaccessible lump we have $V(f, P, \alpha)\Pi_b = \langle \mu \Pi_b \rangle = 0$.

To show that (c) implies (a), let $v = \sum_{b \in B} c_b \mu \Pi_b$ for some coefficients $(c_b)_{b \in B}$ and suppose that $v \in \ker F$. Then

$$0 = vF = \sum_{b \in B} c_b \mu \Pi_b F = \sum_{b \in B} c_b \Big(\sum_{a \in f^{-1}(b)} \mu(a) \Big) e_b.$$

Hence for each $b \in B$, we have $c_b\big(\sum_{a \in f^{-1}(b)} \mu(a)\big) = 0$, and so we have either $\sum_{a \in f^{-1}(b)} \mu(a) = 0$ or $c_b = 0$. In the former case, $\mu(a) = 0$ for every $a \in f^{-1}(b)$ because μ is a non-negative vector. Hence $v = \sum_{b \in B} c_b \mu \Pi_b = 0$, as required. □

Corollary 2.21. *Let P be an irreducible transition matrix and μ its unique stationary probability distribution. Then the following are equivalent:*

- $\mathrm{MC}(\alpha, P)$ *lumps exactly under f,*
- $\mathrm{MC}(\mu, P)$ *lumps exactly under f and $\alpha \Pi_b \propto \mu \Pi_b$ for each $b \in B$,*
- *α belongs to the linear span $\langle \sum_{x \in f^{-1}(b)} \mu(x) e_x : b \in B \rangle$ and this span is preserved by right-multiplication by P.*

Proof. This is immediate from Lemma 2.20. □

Thus in the irreducible case, exact lumping means that if the Markov chain is started in its stationary distribution, then at each later time t, conditional on the history of lumps $f(X_0) = b_0, \ldots, f(X_t) = b_t$, the conditional distribution of X_t on its lump $f^{-1}(\{b_t\})$ is proportional to the restriction of the stationary distribution to that lump.

It appears that the term 'exact lumping' was coined in 1984 by Schweitzer [34] for the special case where α is the uniform distribution on A and α is stationary. It is used in this sense in several later papers, for example [9, 26]. Our definition of exact lumping extends this to more general stationary distributions, and does not require the uniform distribution to be stationary. This more general notion already appeared in 1976 in Kemeny and Snell [24, Thm. 6.4.4] (without the name 'exact') as a sufficient condition for weak lumping in the case where P is irreducible and aperiodic and α is the unique stationary distribution of P. Our definition does not require irreducibility or aperiodicity.

Exact lumping is very closely related to the well-known Pitman–Rogers condition. Suppose $\mathrm{MC}(\alpha, P)$ lumps exactly under f. Since $V(f, P, \alpha)$ is spanned by vectors with non-negative entries, there exists a sub-stochastic matrix U with rows indexed by B and columns indexed by A such that the row of U indexed by b is the unique probability vector in $V(f, P, \alpha)\Pi_b$, if $\dim(V(f, P, \alpha)\Pi_b) = 1$, and is 0 if $\dim(V(f, P, \alpha)\Pi_b) = 0$. Let Q be the transition matrix of the lumped process $f(\mathrm{MC}(\alpha, P))$. Then we have

$$UP = QU. \tag{2.4}$$

This equation is an *algebraic intertwining*, and it is often called the Pitman–Rogers condition, after [30]. The matrix U is often referred to as the *link* matrix. Note that (2.4) does not explicitly mention the initial distribution.

Conversely, suppose that P is a stochastic matrix from A to A and that (2.4) holds for some matrices U and Q, where Q is a stochastic matrix from B to B and U is a matrix from B to A whose rows are either 0 or probability vectors, with the property that $U_{b,a} \neq 0 \implies f(a) = b$. Let α

be any probability vector that is a convex combination of the non-zero rows of U. (Note this requires at least one row of U to be non-zero, ruling out the trivial solution $U = 0$ of (2.4).) Then $V(f, P, \alpha)$ is a subspace of the row span of U and so $\mathrm{MC}(\alpha, P)$ lumps exactly under f. Moreover, Q serves as a transition matrix for the lumped process $f(\mathrm{MC}(\alpha, P))$. Thus the Pitman–Rogers condition implies exact lumping for suitable initial distributions. In the case where P is irreducible, the link matrix U is necessarily stochastic. In fact Pitman and Rogers [30] introduced their intertwining equation as a sufficient condition for weak lumping in the context of continuous time Markov chains; the theory in that case is very similar, with the transition matrices P and Q replaced by generator matrices.

Example 2.22. In the context of a left-invariant random walk on a group G driven by an irreducible weight w, lumping to G/H, the only stationary distribution is the uniform distribution η_G, and so exact lumping may be defined in terms of $V = \langle b\eta_H : bH \in G/H\rangle$. We have $V^\circ = 0$ and $V = \bigoplus_{bH \in G/H} V\Pi_b$ so the only non-trivial condition to check is $VP \subseteq P$. This corresponds to the case $e = \eta_H$ in Theorem 1.2. See Corollary 1.10 for a simpler characterisation of exact lumping in this case.

2.5. **The reversible case.** A stationary Markov chain $X = \mathrm{MC}(\alpha, P)$ with state space A is said to be *reversible* if for all $x, y \in A$ we have

$$\alpha(x)P(x, y) = \alpha(y)P(y, x).$$

This implies that for every $n \geq 0$ the sequence $(X_0, \ldots, X_n)$ has the same joint distribution as its time reversal $(X_n, \ldots, X_0)$. It also means that the chain can be extended to be indexed by times in $\mathbb{Z}$, and this extended chain is equal in law to its own time reversal. The theory of weak lumping simplifies greatly in the reversible case:

Theorem 2.23 (Burke and Rosenblatt (1958) [10, Thm. 1])**.** *Let P be a stochastic matrix from A to A and let α be a stationary distribution for P such that $\alpha(x) > 0$ for all $x \in A$. Suppose that* $\mathrm{MC}(\alpha, P)$ *is reversible and lumps weakly under $f : A \to B$. Then f is a strong lumping of* $\mathrm{MC}(\alpha, P)$.

A close inspection of the proof given in [10] shows that the assumption of full support may be weakened to assuming that α assigns positive weight to each lump of at least two elements.

In §10 we will prove a new duality result, Theorem 10.1, which relates the weak lumping of a stationary finite Markov chain to the weak lumping of its time reversal, under the mild assumption that the stationary distribution has full support. A special case (Corollary 10.2) is that for such stationary Markov chains, time reversal exchanges strong lumping and exact lumping. Applying this to the reversible case, we obtain the following corollary of Theorem 2.23.

Corollary 2.24. *Under the conditions of Theorem 2.23, f is an exact lumping of* $\mathrm{MC}(\alpha, P)$.

2.6. **Probabilistic consequences of strong and exact lumping.** We finish with two results that are not logically required but serve to illuminate the definitions of strong and exact lumping. We maintain our usual notation in which $f : A \to B$ is a surjective function.

Proposition 2.25. *Suppose that* $X = \mathrm{MC}(\alpha, P)$ *lumps strongly under* f. *Then for all* $t \geq 0$, X_t *and* $(f(X_{t+1}), f(X_{t+2}), \ldots)$ *are conditionally independent given* $f(X_t)$.

Proof. For every $y, b \in B$, let $Q(y, b)$ denote the common value of

$$\sum_{a \in f^{-1}(b)} P(x, a)$$

over all $x \in f^{-1}(y)$. The conditional distribution of $(f(X_{t+1}), f(X_{t+2}), \ldots)$ given $f(X_t)$ is determined by its finite-dimensional marginals. From Dynkin's condition, it is easy to show by induction over n that, for each $t \geq 0$, each $x_t \in A$ such that $\mathbb{P}[X_t = x_t] > 0$ and each $n \geq 1$, we have

$$\mathbb{P}[f(X_{t+1}) = y_{t+1}, \ldots, f(X_{t+n}) = y_{t+n} \mid X_t = x_t] = \prod_{i=1}^{n} Q(y_{t+i-1}, y_{t+i}),$$

where $y_t = f(x_t)$. This suffices to demonstrate the conditional independence because the right-hand side depends on x_t only through $f(x_t)$. □

Proposition 2.26. *Suppose that* $X = \mathrm{MC}(\alpha, P)$ *lumps exactly under* f. *Then for all* $t \geq 0$, X_t *and* $(f(X_0), \ldots, f(X_{t-1}))$ *are conditionally independent given* $f(X_t)$.

Proof. Let $V = V(f, P, \alpha)$. Let $b_0, \ldots, b_t$ be any sequence of lumps such that $\mathbb{P}[X_0 = b_0, \ldots, X_t - b_t] > 0$, and let β be the conditional distribution of X_t given $f(X_0) = b_0, \ldots, f(X_t) = b_t$. From the construction of V we have $\beta \in V$. Since β is supported on $f^{-1}(b_t)$, we have $\beta \in V\Pi_{b_t}$. Since $\dim(V\Pi_b) \leq 1$ by Lemma 2.20, β is the unique probability vector in $V\Pi_{b_t}$. In particular, β is a function of b_t alone, which yields the claimed conditional independence. □

3. Background from character theory

We refer the reader to the textbooks of Benson [5], Dornhoff [17] and Isaacs [21] for further background on representation theory and characters; we use the latter two as our basic references in this section. See also [23] for an excellent introduction. Recall that throughout G is a fixed finite group.

3.1. **Group algebras.** The group algebra $\mathbb{C}[G]$ is, by definition, the $|G|$-dimensional vector space of all formal linear combinations $\sum_{g \in G} \beta(g) g$ of the group elements, with coefficients $\beta(g) \in \mathbb{C}$. The multiplication is defined by bilinear extension of the group multiplication: thus

$$\Big(\sum_{x \in G} \beta(x) x\Big)\Big(\sum_{y \in G} \gamma(y) y\Big) = \sum_{g \in G}\Big(\sum_{x \in G} \beta(x)\gamma(x^{-1}g)\Big) g.$$

We identify weights on G with non-zero elements of $\mathbb{C}[G]$ having non-negative real coefficients and probability measures on G with non-negative elements of $\mathbb{C}[G]$ whose coefficient sum is 1. We say that a weight with this property is *normalized*. As the following lemma shows, this makes $\mathbb{C}[G]$ the natural setting for computing with random walks on G.

Lemma 3.1. *In the left-invariant random walk on* G *driven by a normalized weight* w, *if* $X_t \sim \alpha$ *then* $X_{t+1} \sim \alpha w$.

Proof. By definition of the walk

$$\mathbb{P}[X_{t+1} = y] = \sum_{x \in G} \mathbb{P}[X_{t+1} = y | X_t = x]\alpha(x) = \sum_{x \in G} \alpha(x) w(x^{-1}y) = (\alpha w)(y)$$

as required. □

It follows that if w is a normalized weight and $X_0 \sim \alpha$ then $X_t \sim \alpha w^t$ for each $t \geq 0$.

3.2. **Representations, modules and characters.** Let V be a finite-dimensional $\mathbb{C}$-vector space. A *representation* of G is a homomorphism $\rho : G \to \mathrm{GL}(V)$ from G into the general linear group of invertible linear maps on V. We may abuse notation and refer to this representation as V. The group algebra $\mathbb{C}[G]$ then acts on V on the left by

$$\Big(\sum_{g \in G} w(g) g \Big) v = \sum_{g \in G} w(g) \rho(g) v.$$

Thus V becomes a left $\mathbb{C}[G]$-module. (See [17, §1] [21, Definition 1.3] for the definition of modules for an algebra.) Conversely, given a left $\mathbb{C}[G]$-module V, there is a corresponding representation $\rho : G \to \mathrm{GL}(V)$ defined by letting $\rho(g)$ be the linear transformation by which g acts on V. It will often be convenient to pass between these two equivalent languages. The character of a representation $\rho : G \to \mathrm{GL}(V)$, or the corresponding $\mathbb{C}[G]$-module, is the function $\chi_V : G \to \mathbb{C}$ defined by $\chi_V(g) = \operatorname{tr} \rho(g)$, where tr denotes trace. Note that $\chi_V(1) = \dim V$.

Example 3.2. The *trivial representation* of G defined by $\rho(g) = (1) \in \mathrm{GL}_1(\mathbb{C})$ for each $g \in G$ has character $\mathbb{1}_G$, defined by $\mathbb{1}_G(g) = 1$ for each $g \in G$. The *regular representation of* G is the representation afforded by the left action of G on $\mathbb{C}[G]$. Thus in the canonical basis of $\mathbb{C}[G]$ of group elements, each $g \in G$ acts as a $|G| \times |G|$-permutation matrix. It has character ϕ_G defined by $\phi_G(1) = |G|$ and $\phi_G(g) = 0$ for each non-identity $g \in G$.

The regular representation is the case $\Omega = G$ of the following construction.

Example 3.3 (Permutation representations)**.** Suppose that G acts on the left on a set Ω. The *permutation representation of* G *on* Ω, denoted $\mathbb{C}[\Omega]$, has underlying vector space with canonical basis $\{v_\omega : \omega \in \Omega\}$ and action defined by $\rho(g)v_\omega = v_{g\omega}$. If G is a subgroup of Sym_Ω then we say that $\mathbb{C}[\Omega]$ is the *natural* representation of G.

Since we work with left modules, in the action of Sym_Ω on Ω we compose permutations from right to left; thus $(gh)(\omega) = g(h(\omega))$ for $g, h \in \mathrm{Sym}_\Omega$. This convention is in force for the examples in this section only.

Irreducible representations. A *subrepresentation* of a representation $\rho : G \to \mathrm{GL}(V)$ is a subspace W of V such that $\rho(g)W \subseteq W$ for each $g \in G$. A representation V is *irreducible* if it does not contain a non-trivial proper subrepresentation. Representations of finite groups over $\mathbb{C}$ are completely reducible, meaning that one can always write $V = \bigoplus_W W$ where each subrepresentation W is irreducible. (See [17, Theorem 3.1] or [21, Definition 1.7, Theorem 1.9].) We write $\mathrm{Irr}(G)$ for the finite set of irreducible representations of G up to isomorphism, and also for the set of their characters; this creates no ambiguity in practice.

We now begin a running example using the symmetric group Sym_3. By the general theory (see [22, §4]), the irreducible representations of Sym_n are canonically labelled by the partitions of n. Here we give an *ad hoc* construction.

Example 3.4 (Irreducible representations of Sym_3)**.** The natural representation of the symmetric group Sym_3 on $\langle v_1, v_2, v_3\rangle$ decomposes as

$$\langle v_1 + v_2 + v_3\rangle \oplus \langle v_1 - v_3, v_2 - v_3\rangle.$$

The first summand affords the trivial representation of Sym_3 and the second an irreducible 2-dimensional representation. The 1-dimensional sign representation, in which $g \in G$ acts as $\mathrm{sgn}(g) \in \{+1, -1\}$ is the remaining irreducible representation of Sym_3. The corresponding modules are S^3, S^{21} and S^{111}, respectively; we have $S^3 \cong \mathbb{C}$ (the trivial representation) and $S^{111} \cong \mathrm{sgn}$ (the sign representation).

By definition $\mathbb{C}[G]$-modules U and W are *isomorphic* if there is an invertible linear map $T : U \to W$ such that $gT(u) = T(gu)$ for all $u \in U$ and $g \in G$. This holds if and only if U and V have the same character. The character of $U \oplus W$ is $\chi_U + \chi_W$. The number of times an irreducible representation U appears as a summand in a direct sum decomposition of V is $\langle \chi_V, \chi_U\rangle$, where the inner product on characters of G is defined by (1.2). Thus

$$\langle \chi_V, \chi_W\rangle = \frac{1}{|G|}\sum_{g\in G} \overline{\chi_V(g)}\chi_W(g). \tag{3.1}$$

Example 3.5 (Regular representation of Sym_3)**.** The characters of Sym_3 of the irreducible $\mathbb{C}[\mathrm{Sym}_3]$-modules constructed in Example 3.4 are $\chi_{S^3}(g) = 1$, $\chi_{S^{21}}(g) = |\mathrm{Fix}(g)| - 1$ and $\chi_{\mathrm{sgn}}(g) = \mathrm{sgn}(g)$ for each $g \in \mathrm{Sym}_3$. Note $\chi_{S^3} = \mathbb{1}_{\mathrm{Sym}_3}$. The regular character of Sym_3 decomposes as

$$\phi_{\mathrm{Sym}_3} = \chi_{S^3} + 2\chi_{S^{21}} + \chi_{S^{111}}.$$

This is generalized by the Wedderburn decomposition seen in §3.3 following.

As already seen from the regular representation, $\mathbb{C}[G]$ is itself a left $\mathbb{C}[G]$-module. Moreover, a left $\mathbb{C}[G]$-submodule of $\mathbb{C}[G]$ is simply a left ideal in $\mathbb{C}[G]$. Indeed, both are defined as subspaces $L \subseteq \mathbb{C}[G]$ with the property $gL \subseteq L$ for all $g \in G$.

3.3. **Wedderburn decomposition.** Modules for algebras are defined by generalizing the constructions already seen for the group algebra $\mathbb{C}[G]$: see for instance [17, §1]. For our purposes, besides group algebras, the only example we need is $\mathrm{Mat}_d(\mathbb{C})$, the algebra of $d \times d$ matrices. By a basic result (see for instance [17, Theorem 2.18(a)]), $\mathrm{Mat}_d(\mathbb{C})$ has a unique irreducible module up to isomorphism, namely the space of column vectors $\mathbb{C}^n$. Moreover, as a left $\mathrm{Mat}_d(\mathbb{C})$-module,

$$\mathrm{Mat}_d(\mathbb{C}) = W_1 \oplus \cdots \oplus W_d$$

where W_i is the left ideal of $\mathrm{Mat}_d(\mathbb{C})$ of matrices zero except in column i. Each W_i is isomorphic as a $\mathrm{Mat}_d(\mathbb{C})$-module to the irreducible module $\mathbb{C}^n$. If V is a d-dimensional vector space we write $\mathrm{Mat}(V)$ for $\mathrm{Mat}_d(\mathbb{C})$; then V is, up to isomorphism, the unique irreducible module for $\mathrm{Mat}(V)$.

Proposition 3.6 (Wedderburn decomposition)**.** *The group algebra $\mathbb{C}[G]$ admits a decomposition*

$$\mathbb{C}[G] \cong \bigoplus_{V \in \mathrm{Irr}(G)} \mathrm{Mat}(V)$$

as a left $\mathbb{C}[G]$-module and as an algebra, where the sum ranges over a set of representatives of the irreducible representations of G. Moreover the isomorphism may be chosen so that if, in one direct sum decomposition of $\mathbb{C}[G]$ as a left $\mathbb{C}[G]$-module, the $\dim V$ summands isomorphic to the irreducible $\mathbb{C}[G]$-module V are $V_1 \oplus \cdots \oplus V_{\dim V}$, then for each i, the image of V_i in $\mathrm{Mat}(V)$ is the left ideal of $\mathrm{Mat}(V)$ of matrices that are zero except in column i.

Proof. See [17, Theorem 3.2]; this is proved using Theorem 2.18 earlier in [17], from which the 'moreover' part is clear. □

A more condensed proof suitable for experts is given in [5, Theorem 1.3.4]. The proposition is also proved in [21, Theorem 1.15], our reference for character theory, but it takes some work to deduce the version stated above from the subsequent remarks.

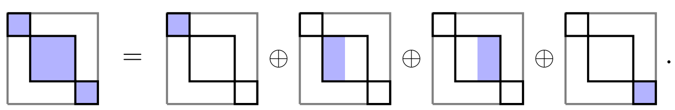
Example 3.7. By Proposition 3.6, the Wedderburn decomposition of Sym_3 into algebra summands is

$$\begin{aligned}\mathbb{C}[\mathrm{Sym}_3] &\cong \mathrm{Mat}_1(\mathbb{C}) \oplus \mathrm{Mat}_2(\mathbb{C}) \oplus \mathrm{Mat}_1(\mathbb{C}) \\ &\cong \mathrm{Mat}(S^3) \oplus \mathrm{Mat}(S^{21}) \oplus \mathrm{Mat}(S^{111})\end{aligned}$$

where, by the 'moreover' part, $\mathrm{Mat}(S^{21}) \cong S^{21} \oplus S^{21}$ as a left $\mathbb{C}[\mathrm{Sym}_3]$-module. A Weddernburn isomorphism, as in this proposition, therefore identifies $\mathbb{C}[\mathrm{Sym}_3]$ with the algebra of 4×4 block matrices

$$\begin{pmatrix} \star & 0 & 0 & 0 \\ 0 & \star & \star & 0 \\ 0 & \star & \star & 0 \\ 0 & 0 & 0 & \star \end{pmatrix}.$$

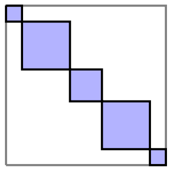
In what comes, we favour the more compact diagrams below, which show a choice of Wedderburn decomposition as in the 'moreover' part.



Example 3.8. The symmetric group Sym_4 has five irreducible representations, labelled by the partitions of 4. The Wedderburn decomposition is

$$\begin{aligned}\mathbb{C}[\mathrm{Sym}_4] &\cong \mathrm{Mat}_1(\mathbb{C}) \oplus \mathrm{Mat}_3(\mathbb{C}) \oplus \mathrm{Mat}_2(\mathbb{C}) \oplus \mathrm{Mat}_3(\mathbb{C}) \oplus \mathrm{Mat}_1(\mathbb{C}) \\ &\cong \mathrm{Mat}(S^4) \oplus \mathrm{Mat}(S^{31}) \oplus \mathrm{Mat}(S^{22}) \oplus \mathrm{Mat}(S^{211}) \oplus \mathrm{Mat}(S^{1111}).\end{aligned}$$

We can therefore identify $\mathbb{C}[\mathrm{Sym}_4]$ with the algebra of 10×10 block matrices of the form



.

We continue this example in Example 3.19.

3.4. **Idempotents.** As we stated in the introduction, an element $e \in \mathbb{C}[G]$ is *idempotent* if $e^2 = e$. It is a basic result that if L is a left ideal in $\mathbb{C}[G]$ (or equivalently, a left $\mathbb{C}[G]$-submodule of $\mathbb{C}[G]$) there exists an idempotent $e \in L$ such that $L = \mathbb{C}[G]e$. Such an idempotent can be constructed by choosing a linear projection $\pi : \mathbb{C}[G] \to L$ and then taking its 'average' $\overline{\pi} = \frac{1}{|G|}\sum_{g \in G} g^{-1}\pi g$. It is routine to check that $\overline{\pi}$ is also a projection onto L, that $\overline{\pi}$ commutes with the action of G, and hence that the map $\overline{\pi}$ agrees with $x \mapsto xe$ where $e = \frac{1}{|G|}\sum_{g \in G} g^{-1}\pi(g) \in L$ is the image of id_G under $\overline{\pi}$. Thus e is a suitable idempotent. We remark that e is not in general unique: see Example 3.18; in fact e is unique if and only if it is a sum of distinct centrally primitive idempotents in the sense defined below.

Example 3.9. The trivial representation of G is isomorphic to the left ideal of $\mathbb{C}[G]$ spanned by the idempotent η_G, defined earlier to be $|G|^{-1}\sum_{g \in G} g$.

Lemma 3.10 (Idempotents versus characters)**.** *Let $e \in \mathbb{C}[G]$ be an idempotent and let W be a $\mathbb{C}[G]$-module. Then $\dim eW = \langle \chi_{\mathbb{C}[G]e}, \chi_W \rangle$.*

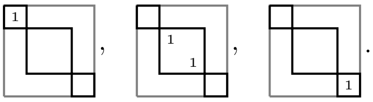
Proof. By complete reducibility we may assume that $\mathbb{C}[G]e$ is isomorphic to the irreducible $\mathbb{C}[G]$-module U. It follows easily from the Wedderburn decomposition that eU is one-dimensional, and $eV = 0$ if V is an irreducible $\mathbb{C}[G]$-module not isomorphic to U. Therefore $\dim eW$ is the number of simple modules isomorphic to U in a direct sum decomposition of W into simple modules; this is the right-hand side. □

By [21, Theorem 2.12] the *centrally primitive idempotent* for an irreducible $\mathbb{C}[G]$ module V with character χ is

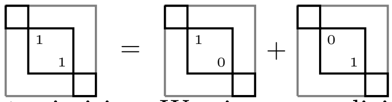
$$e_\chi = \frac{|\chi(1)|}{|G|}\sum_{g \in G} \chi(g^{-1})g. \tag{3.2}$$

(The case where V is the trivial module was seen in Example 3.9.) The image of e_χ in the Wedderburn decomposition is zero except in the block $\mathrm{Mat}(V)$, where it is the identity matrix. It follows easily that $e_\chi L = Le_\chi = L \cap \mathrm{Mat}(V)$ for any left ideal L of $\mathbb{C}[G]$. In general a matrix $e \in \mathrm{Mat}_d(\mathbb{C})$ is an idempotent if and only if it is conjugate to a matrix of the form $\mathrm{diag}(1, ..., 1, 0, ..., 0)$.

Example 3.11 (Centrally primitive idempotents of Sym_3)**.** The centrally primitive idempotents for Sym_3 are $e_3 = \frac{1}{6}\sum_{g \in \mathrm{Sym}_3}(g)$, $e_{21} = \frac{2}{3} - \frac{1}{3}(1,2,3) - \frac{1}{3}(1,3,2)$ and $e_{111} = \frac{1}{6}\sum_{g \in \mathrm{Sym}_3} \mathrm{sgn}(g)g$, where $(1,2,3)$ and $(1,3,2)$ are the two 3-cycles in Sym_3. In any chosen Wedderburn isomorphism they correspond to the identity matrices in the relevant matrix blocks:



A not necessarily central idempotent e is *primitive* if it cannot be expressed as a sum $f + f'$ with f and f' idempotents and $ff' = ff' = 0$. In the previous example, the centrally primitive idempotents e_3 and e_{111} are primitive, since the left ideals that they generate are one-dimensional. Under the Wedderburn decomposition we have

$e_{21} \longmapsto$ 

which shows that e_{21} is *not* primitive. We give an explicit decomposition of e_{21} working in $\mathbb{C}[\mathrm{Sym}_3]$ in Example 3.18 below.

3.5. **Restriction and induction.** Fix throughout a subgroup H of G. We shall often use the restriction and induction functors relating $\mathbb{C}[G]$-modules to $\mathbb{C}[H]$-modules as defined for modules and characters in [17, §9] and for characters in [21, Ch. 5].

Definition 3.12 (Restricted and induced modules)**.**
(a) The *restriction of a* $\mathbb{C}[G]$*-module* W *to* H, denoted $W\downarrow_H^G$, is the $\mathbb{C}[H]$-module with the same underlying vector space as W, but the action defined only on H.
(b) The *induction of a* $\mathbb{C}[H]$*-module* U *to* G, denoted $U\uparrow_H^G$, is defined to be $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} U$ where $\mathbb{C}[G]$ is regarded as a right $\mathbb{C}[H]$-module by right multiplication.

In (b), the space $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} U$ may be defined as the quotient of $\mathbb{C}[G] \otimes U$ by the subspace spanned by all $gh \otimes u - g \otimes hu$ for $x \in G$, $h \in H$ and $u \in U$. Thus the relation $gh \otimes u = x \otimes hu$ holds in $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} U$. This vector space is a $\mathbb{C}[G]$-module with action defined by linear extension of $k(g \otimes u) = kg \otimes u$ for $k \in G$. In many cases one can avoid thinking about the technical construction using tensor products and instead employ the following characterisation.

Proposition 3.13 (Characterisation of induced modules)**.** *Let* U *be a* $\mathbb{C}[H]$*-module. The following are equivalent for a* $\mathbb{C}[G]$*-module* V*:*
(i) $V \cong U\uparrow_H^G$*;*
(ii) V *has a* $F[H]$*-submodule* X *isomorphic to* U *such that* X *generates* V *as a* $\mathbb{C}[G]$*-module and* $\dim V = [G : H] \dim X$*;*
(iii) V *has a* $F[H]$*-submodule* X *isomorphic to* U *and there is a vector space decomposition* $V = \bigoplus_{g \in G/H} gX$.

Proof. For the equivalence of (i) and (ii), see [2, page 56, Corollary 3]. By dimension counting one sees that that (ii) implies (iii) and the converse is obvious. □

Example 3.14. Let $\Omega = G/H$. The permutation module $\mathbb{C}[\Omega]$ of G acting on the cosets of H, as defined in Example 3.3, has as a canonical basis

$\{v_{bH} : b \in G/H\}$. Observe that $X = \langle v_H \rangle$ affords the trivial representation $\mathbb{C}$ of H and that $\dim \mathbb{C}[\Omega] = |G/H|$. Therefore by Proposition 3.13 we have $\mathbb{C}[\Omega] \cong \mathbb{C}\!\uparrow_H^G$.

In particular, if U is a left ideal of $\mathbb{C}[H]$ then by Proposition 3.13(iii) the left ideal of $\mathbb{C}[G]$ generated by U, namely $\bigoplus_{g \in G/H} gU$, is isomorphic to $U\!\uparrow_H^G$. Such ideals are ubiquitous in this work. Given a left coset $bH \in G/H$, Let $\pi_{bH} : \mathbb{C}[G] \to \mathbb{C}[H]$ denote the projection map defined by

$$\pi_{bH}\Bigl(\sum_{g \in G} x(g)g\Bigr) = \sum_{g \in bH} x(g)g.$$

Definition 3.15. We say that an ideal L of $\mathbb{C}[G]$ containing $\pi_H(L)$ is an *induced ideal* from H to G.

We typically omit 'from H to G' as it is clear from context.

Example 3.16. Observe that $\langle \eta_H \rangle$ is a left ideal of $\mathbb{C}[H]$ affording the trivial representation of H. Therefore, by the remark before Definition 3.15,

$$\langle \eta_H \rangle \big\uparrow_H^G \cong \mathbb{C}[G]\eta_H = \langle b\eta_H : b \in G/H \rangle.$$

Working directly from Definition 3.12(ii) one could instead show that $g \otimes \eta_H \to g\eta_H$ defines a $\mathbb{C}[G]$-isomorphism $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} \langle \eta_H \rangle \cong \mathbb{C}[G]\eta_H$; this is a routine, but somewhat technical, check.

The following proposition justifies the name 'induced ideal' and generalizes the features seen in the previous example.

Proposition 3.17. *Let L be a left ideal of $\mathbb{C}[G]$ containing $\pi_H(L)$. Setting $U = \pi_H(L)$, we have*

(i) *U is a left ideal of $\mathbb{C}[H]$;*
(ii) *$L = \mathbb{C}[G]U$;*
(iii) *$L = \bigoplus_{b \in G/H} bU$;*
(iv) *there is an isomorphism of left $\mathbb{C}[G]$-modules $L \cong U\!\uparrow_H^G$;*
(v) *there exists an idempotent $e \in \mathbb{C}[H]$ such that $L = \mathbb{C}[G]e$.*

Proof. Since L is a left ideal of $\mathbb{C}[G]$ we have $\mathbb{C}[H]U \subseteq L$ and since $\mathbb{C}[H]$ is closed under multiplication by elements of H, we have $\mathbb{C}[H]U \subseteq \mathbb{C}[H]$. Therefore $\mathbb{C}[H]U \subseteq L \cap \mathbb{C}[H] = U$, proving (i). Since L is a left ideal of $\mathbb{C}[G]$, L contains $\bigoplus_{b \in G/H} bU$, and since

$$L \cap b\mathbb{C}[H] \subseteq b(L \cap \mathbb{C}[H]) = bU,$$

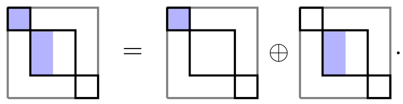
it follows that $L = \bigoplus_{b \in G/H} bU$. This proves (ii) and (iii); now (iv) follows from Proposition 3.13(iii). Finally by the remark at the start of §3.4, applied to the left ideal U of $\mathbb{C}[H]$, there exists an idempotent $e \in \mathbb{C}[H]$ such that $U = \mathbb{C}[H]e$. It now follows easily from (iii) that $L = \mathbb{C}[G]e$. □

Example 3.18 (Primitive idempotents of Sym_3)**.** We remarked after Example 3.11 that the centrally primitive idempotents $e_3 = \eta_{\mathrm{Sym}_3}$ and $e_{111} = \frac{1}{6}\sum_{g \in \mathrm{Sym}_3} \mathrm{sgn}(g)g$ are primitive. Let $H = \mathrm{Sym}_{\{2,3\}}$. The natural action of Sym_3 on $\{1, 2, 3\}$ corresponds to the action of Sym_3 on the left cosets of H. (More formally, the map $gH \mapsto g(1)$ is a permutation isomorphism.) By Example 3.16, the natural permutation representation $\langle v_1, v_2, v_3 \rangle$ for Sym_3

(seen earlier in Example 3.4) is isomorphic to the left ideal $\mathbb{C}[G]\eta_H$, by an isomorphism satisfying $v_1 \mapsto \eta_H$. We saw earlier that $\langle v_1, v_2, v_3\rangle \cong S^3 \oplus S^{21}$ where S^3 is the trivial module. Therefore subtracting η_{Sym_3} from η_H we obtain

$$f = \eta_H - \eta_{\mathrm{Sym}_3} = \tfrac{1}{3}\mathrm{id}_{\mathrm{Sym}_3} - \tfrac{1}{6}(1,2,3) - \tfrac{1}{6}(1,3,2) - \tfrac{1}{6}(1,2) - \tfrac{1}{6}(1,3) + \tfrac{1}{3}(2,3).$$

Since η_{Sym_3} is central, f is an idempotent such that $\mathbb{C}[\mathrm{Sym}_3]f \cong S^{21}$. Taking $f' = e_{21} - f$ gives an explicit decomposition of the central primitive idempotent e_{21} into primitive idempotents, as promised after Example 3.11. It is worth noting that there is nothing canonical about this decomposition: indeed f may be replaced with any conjugate ufu^{-1} for a unit $u \in \mathbb{C}[G]$. The images of η_H, η_G and $f = \eta_H - \eta_G$ are shown below, with respect to the Wedderburn decomposition $\mathbb{C}[\mathrm{Sym}_3] = \mathbb{C}[\mathrm{Sym}_3]\eta_G + \big(\mathbb{C}[\mathrm{Sym}_3]f + \mathbb{C}[\mathrm{Sym}_3]f'\big) + \mathbb{C}[\mathrm{Sym}_3]e_{111}$:



Definition 3.12 extends to characters and representations in the obvious way: let $\chi_W\!\downarrow_H^G$ be the character of $W\!\downarrow_H^G$ and let $\chi_U\!\uparrow_H^G$ be the character of $U\!\uparrow_H^G$.

Example 3.19. Let $H = \mathrm{Sym}_3$ and $G = \mathrm{Sym}_4$. Refer to Examples 3.4, 3.7 and 3.8. We have

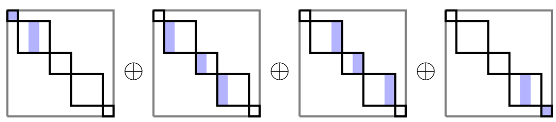
(1) $\chi^3\!\uparrow_{\mathrm{Sym}_3}^{\mathrm{Sym}_4} = \chi^4 + \chi^{31}$,
(2) $\chi^{21}\!\uparrow_{\mathrm{Sym}_3}^{\mathrm{Sym}_4} = \chi^{31} + \chi^{22} + \chi^{211}$, and
(3) $\chi^{111}\!\uparrow_{\mathrm{Sym}_3}^{\mathrm{Sym}_4} = \chi^{211} + \chi^{1111}$.

In each case the character of Sym_4 is obtained by adding a box to the relevant partition of 3, in all possible ways. (See [22, Ch. 9] for the general result.) Corresponding to the Wedderburn decomposition

$$\mathbb{C}[\mathrm{Sym}_3] \cong S^3 \oplus S^{21} \oplus S^{21} \oplus S^{111}$$

from Example 3.7, there is a choice of Wedderburn isomorphism for $\mathbb{C}[\mathrm{Sym}_4]$ such that

$$\mathbb{C}[\mathrm{Sym}_4] \cong \mathbb{C}[\mathrm{Sym}_3]\big\uparrow_{\mathrm{Sym}_3}^{\mathrm{Sym}_4} \cong$$



where the four summands are the modules induced from the summands of $\mathbb{C}[\mathrm{Sym}_3]$ displayed above and the blocks are ordered from top-left to bottom-right 4, 31, 22, 211, 1111. This isomorphism is chosen so that tensoring by the sign representation S^{1111} corresponds to rotating diagrams by a half-turn; note that $S^{31} \otimes \mathrm{sgn} \cong S^{211}$ and $S^{22} \otimes \mathrm{sgn} \cong S^{22}$. We return to this example in Example 7.5.

We end this subsection with a fundamental result relating induced and restricted modules and characters. Below the subscripts G and H indicate the group relevant to the character inner product defined in (3.1).

Proposition 3.20 (Frobenius reciprocity). *If U is a $\mathbb{C}[H]$-module and W is a $\mathbb{C}[G]$-module then*

$$\big\langle \chi_U\big\uparrow_H^G, \chi_W\big\rangle_G = \big\langle \chi_U, \chi_W\big\downarrow_H^G\big\rangle_H.$$

Proof. See [17, Theorem 9.4(c)], [21, Lemma 5.2] for proofs with minimal prerequisites, or, for an elegant and conceptual proof using the tensor-hom adjunction, [5, Proposition 2.8.3]. □

3.6. **Borel and parabolic subalgebras.** We continue with some basic results on subalgebras of $\mathrm{Mat}_d(\mathbb{C})$ needed in the proof of Proposition 1.8. Proofs are included to make the article self-contained and to introduce some ideas relevant to the proof of this proposition. Recall that a subalgebra P of $\mathrm{Mat}_d(\mathbb{C})$ is *parabolic* if there is a chain of subspaces $\mathbb{C}^d \supset V_1 \subset \ldots \supset V_r \supset 0$ such that $P = \{T \in \mathrm{Mat}_d(\mathbb{C}) : T(V_i) \subseteq V_i \text{ for } 1 \le i \le r\}$. In this case we write

$$P = \mathrm{Stab}(\mathbb{C}^d \supset V_1 \supset \ldots \supset V_r \supset 0).$$

Observe that if M is an invertible matrix then

$$MPM^{-1} = \mathrm{Stab}\big(\mathbb{C}^d \supset M(V_1) \supset \ldots \supset M(V_r) \supset 0\big). \qquad (3.3)$$

Stated slightly informally, the following lemma is that parabolic subalgebras are self-normalizing.

Lemma 3.21. *Let P be a parabolic subalgebra of $\mathrm{Mat}_d(\mathbb{C})$. Let M be an invertible matrix in $\mathrm{Mat}_d(\mathbb{C})$. If $MPM^{-1} = P$ then $M \in P$.*

Proof. Let $P = \mathrm{Stab}(\mathbb{C}^d \supset V_1 \supset \ldots \supset V_r \supset 0)$ be a parabolic subalgebra of $\mathrm{Mat}_d(\mathbb{C})$. Observe that V_1 is the greatest proper subspace of $\mathbb{C}^d$ preserved by P, and inductively, V_i is the greatest proper subspace of V_{i-1} preserved by P, for each i. Thus P determines the chain of subspaces $\mathbb{C}^d \supset V_1 \supset \ldots \supset V_r \supset 0$. It now follows from (3.3) that $MPM^{-1} = P$ if and only if $M(V_i) = V_i$ for each i, or equivalently, if and only if M is an invertible matrix in P. The lemma follows. □

We define the *standard Borel subalgebra* of $\mathrm{Mat}_d(\mathbb{C})$ to be its subalgebra of invertible lower triangular matrices. Equivalently, if C_i is the subspace of $\mathbb{C}^d$ of column vectors zero in their top $d-i$ positions, then the standard Borel subalgebra is $\mathrm{Stab}(\mathbb{C}^d \supset C_{d-1} \supset \ldots \supset C_1 \supset 0)$. We say that a subalgebra of $\mathrm{Mat}_d(\mathbb{C})$ is *Borel* if it is conjugate by an invertible matrix to the standard Borel. We say that a parabolic subalgebra of $\mathrm{Mat}_d(\mathbb{C})$ is *standard* if it contains the standard Borel subalgebra.

Lemma 3.22. *A parabolic subalgebra is standard if and only if it is equal to $\mathrm{Stab}(\mathbb{C}^d \supset C_{d_1} \supset \ldots \supset C_{d_r} \supset 0)$ for some $d > d_1 > \ldots > d_r > 0$.*

Proof. The 'if' direction is clear from the equivalent definition of the standard Borel algebra just given. For the 'only if' direction, let $P = \mathrm{Stab}(\mathbb{C}^d \supset V_1 \supset \ldots \supset V_r \supset 0)$ be a standard parabolic subalgebra. Let B be its subgroup of invertible lower triangular matrices. Observe that the orbits of B on $\mathbb{C}^d$ are $\{0\}$ and $\mathcal{O}_i$ for $1 \le i \le d$, where

$$\mathcal{O}_i = \{v \in \mathbb{C}^d : v_1 = \ldots = v_{i-1} = 0, v_i \neq 0\}.$$

If V is a subspace of $\mathbb{C}^d$ such that $P(V) \subseteq V$ then since $B(V) = V$, the subspace V is a union of orbits of these orbits. Applying this observation to each V_i in turn we find that $V_i = C_{d_i}$ where $d_i = \dim V_i$, for each $1 \le i \le r$. Hence $P = \mathrm{Stab}(\mathbb{C}^d \supset C_{d_1} \supset \ldots \supset C_{d_r} \supset 0)$ as required. □

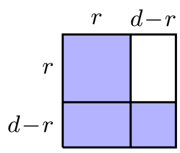
Proposition 3.23. *Each parabolic subalgebra of* $\mathrm{Mat}_d(V)$ *is conjugate to a unique standard parabolic.*

Proof. Let $P = \mathrm{Stab}(\mathbb{C}^d \supset V_1 \supset \ldots \supset V_r \supset 0)$ be a parabolic subalgebra of $\mathrm{Mat}_d(\mathbb{C})$. Starting with V_r and working backwards, one may construct an invertible matrix M such that $M(V_i) = C_{\dim V_i}$ for each $1 \le i \le r$. By (3.3) we have $MPM^{-1} = \mathrm{Stab}(\mathbb{C}^d \supset C_{\dim V_1} \supset \ldots \supset C_{\dim V_r} \supset 0)$. By the 'if' direction of Lemma 3.22, MPM^{-1} is a standard parabolic. By (3.3), the dimensions of the V_i are preserved by conjugacy. Therefore MPM^{-1} is the unique standard parabolic conjugate to P. □

3.7. **Right idealizers.** We finish our algebraic background with a result on idealizers needed in the proof of Theorem 1.2. Given a left ideal L in $\mathbb{C}[G]$ its *right idealizer* $\mathrm{RId}_{\mathbb{C}[G]}(L)$ is the largest subspace W of $\mathbb{C}[G]$ such that L is a right ideal in W. Note that since L is closed under multiplication, W contains L. In symbols,

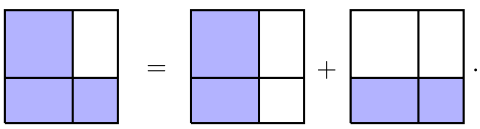
$$\mathrm{RId}_{\mathbb{C}[G]}(L) = \{w \in \mathbb{C}[G] : Lw \subseteq L\}.$$

Lemma 3.24. *Let* $e \in \mathbb{C}[G]$ *be an idempotent, let* $L = \mathbb{C}[G]e$ *be a left ideal of* $\mathbb{C}[G]$*. Then* $\mathrm{RId}_{\mathbb{C}[G]}(L) = \mathbb{C}[G]e + (1-e)\mathbb{C}[G]$*. Moreover, the Wedderburn component of* $\mathrm{RId}_{\mathbb{C}[G]}(L)$ *in the block* $\mathrm{Mat}(V)$ *is (up to a choice of isomorphism) the parabolic subalgebra* $\mathrm{Stab}(\mathbb{C}^d \supset C_r \supset \mathbb{C}^d)$

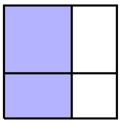
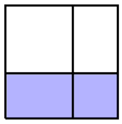


where r is the multiplicity of the irreducible module V in L and $d = \dim(V)$.

After this lemma, the formula $\mathrm{RId}_{\mathbb{C}[G]}(L) = \mathbb{C}[G]e + (1-e)\mathbb{C}[G]$ can be represented (on each Wedderburn block) as



Proof of Lemma 3.24. By the Wedderburn decomposition in Proposition 3.6, we can write $e = \sum_{V \in \mathrm{Irr}(G)} e_V$ in a unique way, where e_V is the part of e supported on the Wedderburn block $\mathrm{Mat}(V)$. Moreover, since each Wedderburn block is an algebra, we deduce that e_V is an idempotent for every $V \in \mathrm{Irr}(G)$. Up to a choice of Wedderburn isomorphism, we can identify each e_V with the diagonal matrix $\mathrm{diag}(1, ..., 1, 0, ..., 0)$ with exactly $r = \langle \chi_{\mathbb{C}[G]e}, \psi_V \rangle$ ones. This gives

$\mathbb{C}[G]e =$  and $(1-e)\mathbb{C}[G]e =$  .

Checking the claims is now a linear algebra exercise. □

Remark 3.25. Note that the right idealizer of a left ideal and its normalizer are closely related. Indeed, let L be a left ideal. Then working in the group $\mathbb{C}[G]^\times$ of invertible elements of $\mathbb{C}[G]$, we have

$$N_{\mathbb{C}[G]^\times} L = \{w \in \mathbb{C}[G]^\times : w^{-1}Lw = L\} = \{w \in \mathbb{C}[G]^\times : Lw = L\} = \mathrm{RId}_{\mathbb{C}[G]}(L) \cap \mathbb{C}[G]^\times$$

where the second equality holds because L is a left ideal and the third because $Lw = L$ if and only if $Lw \subseteq L$ for invertible elements w.

4. Double cosets

In this section we collect some basic results on double cosets, giving a short algebraic proof of a key 'averaging' lemma and a special case of Mackey's restriction formula. Fix throughout this section a finite group G and subgroups T and H of G. By definition, the double coset TxH is the set $\{txh : t \in T, h \in H\}$.

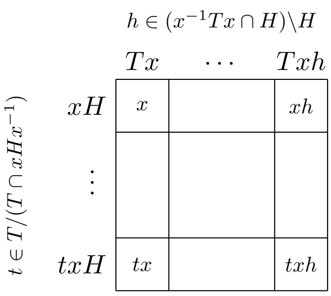
4.1. **Counting results.** It is clear that TxH is a union of left cosets of H, and also a union of right cosets of T. The different expressions for elements of TxH all come from the equation

$$txh = tsxs'h \tag{4.1}$$

where $s \in T \cap xHx^{-1}$ and so $s' = x^{-1}s^{-1}x \in xTx^{-1} \cap H$. It follows that

$$\begin{aligned} TxH &= \{txh : t \in T, h \in (x^{-1}Tx \cap H)\backslash H\} \\ &= \{txh' : t \in t/(T \cap xHx^{-1}), h \in H\}. \end{aligned}$$

Thus TxH is a disjoint union of the right cosets Txh for h in a set of representatives for the right cosets of $x^{-1}Tx \cap H$ in H, and also a disjoint union of the left cosets txH for t in a set of representatives for the left cosets of $T \cap xHx^{-1}$ in T. This is shown diagrammatically below, using the identity as one coset representative in each case.



By (4.1), each box has $|T \cap xHx^{-1}| = |x^{-1}Tx \cap H|$ different elements of HxH. Denoting this common value by r, there are $|T|/r$ rows and $|H|/r$

columns. By counting elements we obtain the equation $r|T||H|/r^2 = |TxH|$, or equivalently

$$|TxH|\,|x^{-1}Tx \cap H| = |H|\,|T|. \tag{4.2}$$

As an immediate application of (4.2) we prove the following key 'averaging' lemma. Recall from the start of §1.1 that if $K \leq G$ is a subgroup then $\eta_K = |K|^{-1} \sum_{k \in K} k \in \mathbb{C}[G]$, where $\mathbb{C}[G]$ is the group algebra defined in §3.1.

Lemma 4.1. *For $w \in \mathbb{C}[G]$ we have*

(i) *$\eta_T w$ is constant on each right coset Tg in TxH and its common value on Tx is $w(Tx)/|T|$;*
(ii) *$w\eta_H$ is constant on each left coset gH in HxH and its common value on xH is $w(xH)/|H|$;*
(iii) *$\eta_T w \eta_H$ is constant on TxH and its common value on the double coset is $w(TxH)/|TxH|$.*

Proof. We have $(\eta_T wv)(x) = |T|^{-1} \sum_{t \in T} w(tx)$; this is $w(Tx)/|T|$, as required for (i). The proof of (ii) is dual. By (i) and (ii) each weight in $\eta_T \mathbb{C}[G] \eta_H$ is constant on TxH. Let the common value of $\eta_T w \eta_H$ be c. By (i), then (ii), then (4.2), we have

$$c = \frac{(w\eta_H)(Tx)}{|T|} = \frac{1}{|T||H|} \sum_{h \in H} w(Txh) = \frac{|x^{-1}Tx \cap H|}{|T||H|} w(TxH) = \frac{w(TxH)}{|TxH|}$$

as required. □

We outline an alternative proof of (iii) that probability-minded readers will find very intuitive: one can sample uniformly at random from TxH by choosing $t \in T$ and $h \in H$ according to the distributions η_T and η_H and then taking txh. Hence the expected value of w on TxH, namely $w(TxH)/|TxH|$, is the common value of $\eta_T w \eta_H$ on TxH.

4.2. **Mackey's rule for the trivial representation.** We have seen that the orbits of T acting on the left on the set G/H of left cosets are each of the form $\{txH : t \in T\}$, and that the stabiliser of the distinguished orbit representative xH is $T \cap xHx^{-1}$.

Lemma 4.2 (Mackey's rule for the trivial representation)**.** *Let $\mathbb{C}$ be the trivial representation of H. There is an isomorphism of left $\mathbb{C}[H]$-modules*

$$\mathbb{C}{\uparrow}_H^G{\downarrow}_H \cong \bigoplus_{x \in H\backslash G/H} \mathbb{C}{\uparrow}_{H \cap xHx^{-1}}^H,$$

where the sum ranges over a set of double coset representatives for $H\backslash G/H$.

Proof. Taking $T = H$ in the remark before the lemma, it follows that the $\mathbb{C}[H]$-permutation module of H acting transitively on its orbit $\{hxH : h \in H\}$ of left cosets is $\mathbb{C}{\uparrow}_{H \cap xHx^{-1}}^H$. □

This result is generalized by Lemma 12.4 below.

5. A characterisation of weak lumping of left-invariant random walks

Recall that in our standing notation H is a fixed subgroup of the finite group G. In this section we work throughout with an irreducible weight w and characterise the initial distributions α such that the left-invariant random walk driven by w with initial distribution α lumps weakly to the left cosets G/H, in the sense of Definition 1.1(a).

5.1. The minimal Gurvits–Ledoux space and minimal Gurvits–Ledoux left ideal.

Recall from before Definition 3.15 that if $bH \in G/H$ then $\pi_{bH} : \mathbb{C}[G] \to \mathbb{C}[G]$ is the projection map onto the coset bH, defined by $\pi_{bH}\big(\sum_{g\in G} x(g)g\big) = \sum_{g\in bH} x(g)g$. By Definition 2.5, the minimal complex Gurvits–Ledoux vector space for the left-invariant random walk driven by w started at the probability distribution $\alpha \in \mathbb{C}[G]$ is the intersection of all vector subspaces V of $\mathbb{C}[G]$ satisfying

- (V0) $\alpha \in V$,
- (V1) $Vw \subseteq V$,
- (V2) $\pi_{bH}(V) \subseteq V$ for all $bH \in G/H$.

Note this intersection is well-defined as $\mathbb{C}[G]$ satisfies (V0), (V1) and (V2) and each of the conditions is closed under intersection. We denote this minimal vector space by $V_{\alpha,w}$. To compare with the definitions in §2.1, $V_{\alpha,w}$ can be identified with $\mathbb{C}\otimes_{\mathbb{R}} V(f, P, \alpha)$ where P is the transition matrix of the left-invariant walk driven by w and f is the left coset mapping $G \to G/H$. Note that $V_{\alpha,w}$ is a complex vector subspace of the group algebra $\mathbb{C}[G]$.

Recall that if T is a non-empty subset of G then $\eta_T = |T|^{-1} \sum_{t\in T} t \in \mathbb{C}[G]$.

Lemma 5.1. *Let w be an irreducible weight. If V is a subspace of $\mathbb{C}[G]$ satisfying* (V0)*,* (V1) *and* (V2) *then $\eta_G \in V$.*

Proof. Since w is irreducible, the unique stationary distribution of the left-invariant random walk is $\eta_G \in \mathbb{C}[G]$. As in Definition 2.5, V is a complex Gurvits–Ledoux space containing a probability vector α, so $V \supseteq \mathbb{C} \otimes_{\mathbb{R}} V(f, P, \alpha)$. By Lemma 2.13, $\eta_G \in V(f, P, \alpha)$. □

Proposition 5.2. *If w is an irreducible weight then $V_{\eta_G,w} \subseteq V_{\alpha,w}$.*

Proof. If V is a subspace of $\mathbb{C}[G]$ satisfying (V0), (V1) and (V2) *for* α then, by Lemma 5.1, V satisfies (V0) *for* η_G, and hence V contains the minimal complex Gurvits–Ledoux space $V_{\eta_G,w}$. The proposition follows by taking the intersection over all such V. □

Lemma 5.3. *The subspace $V_{\eta_G,w}$ is a left ideal of $\mathbb{C}[G]$. Moreover, $\pi_H(V_{\eta_G,w})$ is a left ideal of $\mathbb{C}[H]$ and $V_{\eta_G,w} = \mathbb{C}[G]\big(\pi_H(V_{\eta_G,w})\big)$ is an induced ideal.*

Proof. For readability, let $L = V_{\eta_G,w}$. To show that L is a left ideal it suffices to show that $k^{-1}L \supseteq V_{\eta_G,w}$ for each $k \in G$, since by multiplying by k we then obtain $L \supseteq kL$. Setting $g = k^{-1}$ we therefore check that gL satisfies (V0)–(V2). We may then deduce $gL \supseteq L$ by minimality of L:

- (V0) $g\eta_G = \eta_G \in L$ by (V0) for L;
- (V1) $(gL)w \subseteq gL$ since $Lw \subseteq L$ by (V1) for L;
- (V2) since $\pi_{bH}g = g\pi_{g^{-1}bH}$, we have $\pi_{bH}(gL) = g\pi_{g^{-1}bH}(L) \subseteq gL$ by (V2) for L.

Hence L is a left ideal of $\mathbb{C}[G]$. Again by (V2) for L, we have $\pi_H(L) \subseteq L$. Therefore by Proposition 3.17(ii), $L = \mathbb{C}[G]\big(\pi_H(L)\big)$. $\square$

Thus $V_{\eta_G,w}$, which *a priori* was merely a subspace of $\mathbb{C}[G]$, is in fact an induced ideal in the sense of Definition 3.15.

5.2. **Application of the Gurvits–Ledoux characterisation to prove Corollary 1.6.** Let $\mathbb{C}^{G/H}$ be the vector space of complex valued functions on G/H and let $\Lambda : \mathbb{C}[G] \to \mathbb{C}^{G/H}$ be the linear map induced by the canonical quotient map $G \mapsto G/H$ defined by $g \mapsto gH$.

Lemma 5.4. $\ker \Lambda = \mathbb{C}[G](1 - \eta_H)$.

Proof. We have $x \in \ker \Lambda$ if and only if $\sum_{g \in bH} x(g) = 0$ for each $b \in G$, and so if and only if $x \in \bigoplus_{b \in G/H} b\mathbb{C}[H](1 - \eta_H) = \mathbb{C}[G](1 - \eta_H)$. $\square$

Given a vector subspace V of $\mathbb{C}[G]$, let V° denote the subspace $V \cap \ker \Lambda$, in accordance with (2.1). We introduce a further property:

(V3) $V^\circ w \subseteq V^\circ$.

Since $\ker \Lambda$ and $V_{\alpha,w}$ have bases consisting of real vectors, $V^\circ_{\alpha,w}$ also has a basis consisting of real vectors, which span the real vector space $V(f, P, \alpha)^\circ$. Moreover, $V(f, P, \alpha)^\circ P \subseteq V(f, P, \alpha)^\circ$ if and only if $V^\circ_{\alpha,w} w \subseteq V^\circ_{\alpha,w}$. Thus in this setting the conclusion of Theorem 2.6 may be written as follows:

$$\mathrm{MC}(\alpha, w) \text{ lumps weakly to } G/H \text{ if and only if } V_{\alpha,w} \text{ satisfies (V3).} \qquad (\star)$$

We use ($\star$) to prove Corollary 1.6. The following lemmas are required.

Lemma 5.5. *Let w be an irreducible weight. If* (V3) *holds for $V_{\alpha,w}$ then* (V3) *holds for $V_{\eta_G,w}$.*

Proof. We have $V^\circ_{\eta_G,w} w \subseteq V_{\eta_G,w} w \subseteq V_{\eta_G,w}$. By Proposition 5.2 we have $V_{\eta_G,w} \subseteq V_{\alpha,w}$. Hence

$$V^\circ_{\eta_G,w} w = (V_{\eta_G,w} \cap \ker \Lambda) w \subseteq (V_{\alpha,w} \cap \ker \Lambda) w = V^\circ_{\alpha,w} w \subseteq V^\circ_{\alpha,w} \subseteq \ker \Lambda$$

and so $V^\circ_{\eta_G,w} w \subseteq V_{\eta_G,w} \cap \ker \Lambda = V^\circ_{\eta_G,w}$. $\square$

Lemma 5.6. *Let V be a subspace of $\mathbb{C}[G]$ satisfying* (V2)*. Suppose that $\eta_G \in V$. Then $V^\circ = V(1 - \eta_H)$.*

Proof. By (V2), $V = \bigoplus_{b \in G/H} V \cap b\mathbb{C}[H]$ where $V \cap b\mathbb{C}[H] = \pi_{bH}(V)$. By (V0) and (V2) we have $\pi_H(\eta_G) = \eta_H \in V$. Given $u \in b\mathbb{C}[H]$ we have $u\eta_H \in \langle b\eta_H \rangle$, hence each subspace $\pi_{bH}(V)$ is closed under right multiplication by the idempotent $1 - \eta_H$. Now using $V^\circ = V \cap \ker \Lambda$ and Lemma 5.4 we have

$$V^\circ = \Big(\bigoplus_{b \in G/H} \pi_{bH}(V) \Big) \cap \Big(\bigoplus_{b \in G/H} b\mathbb{C}[H](1 - \eta_H) \Big) = \bigoplus_{b \in G/H} \pi_{bH}(V)(1 - \eta_H).$$

Therefore $V^\circ = V(1 - \eta_H)$, as required. $\square$

Lemma 5.7. *If L is an induced left ideal from H to G containing η_G then $L^\circ = L(1 - \eta_H)$.*

Proof. By Proposition 3.17(iii), L satisfies (V2). Now apply Lemma 5.6. $\square$

For ease of notation, from now on we shall write L_w for $V_{\eta_G,w}$; note that by Lemma 5.3, L_w is a left ideal of $\mathbb{C}[G]$, so this is consistent with our usual notational conventions.

Corollary 1.6 (Weak lumping test for a weight). *Let w be an irreducible weight. The following are equivalent:*

(i) *The left-invariant random walk driven by w lumps weakly to G/H;*
(ii) $\mathrm{MC}(\eta_G, w)$ *lumps weakly to G/H;*
(iii) $L_w(1-\eta_H)w\eta_H = 0$*;*
(iv) *The left-invariant random walk driven by w lumps to G/H with stable ideal L_w.*

Proof of Corollary 1.6. Suppose that (i) holds, so there exists a starting distribution α such that $\mathrm{MC}(\alpha, w)$ lumps weakly to G/H. Then, by ($\star$), the minimal Gurvits–Ledoux vector space $V_{\alpha,w}$ satisfies (V3). By Proposition 5.2, using our hypothesis that w is irreducible, L_w is contained in $V_{\alpha,w}$. By Lemma 5.5, L_w also satisfies (V3). Hence, by the 'if' direction of ($\star$), $\mathrm{MC}(\eta_G, w)$ lumps weakly to G/H, proving (ii).

By the 'only if' direction of ($\star$), (ii) implies in particular that L_w satisfies (V3), that is, $L_w^\circ w \subseteq L_w$. By Lemma 5.3, L_w is an induced ideal and so by Lemma 5.7, $L_w^\circ = L_w(1-\eta_H)$. Therefore $L_w(1-\eta_H)w \subseteq L_w(1-\eta_H)$. In particular

$$L_w(1-\eta_H)w\eta_H \subseteq L_w(1-\eta_H)\eta_H = 0,$$

where the final equality holds because $(1-\eta_H)\eta_H = 0$. This proves (iii) and shows that (iii) is equivalent to $L_w^\circ w \subseteq L_w^\circ$. This is condition (d) in Definition 2.10, and conditions (a), (b) and (c) in this definition hold by (V0), (V1), (V2) for L_w. Therefore (iii) implies (iv), namely that the left-invariant random walk driven by w lumps stably for the left ideal L_w. In particular, the left-invariant random walk lumps weakly when started at η_G, which implies (i) on taking $\alpha = \eta_G$. □

5.3. **The maximal Gurvits–Ledoux ideal J_w.** Playing an equally important role to the induced left ideal $L_w = V_{\eta_G,w}$, which by Proposition 5.2 should be thought of as the *minimal* Gurvits–Ledoux space for η_G and w, we will shortly define the *maximal* Gurvits–Ledoux ideal J_w. The following two lemmas are implied by Lemma 2.15 but for the reader's convenience we give short proofs here in the language of group algebras. Recall property (V2) of a subspace V of $\mathbb{C}[G]$ is that $\pi_{bH}(V) \subseteq V$ for all $bH \in G/H$.

Lemma 5.8. *Let V and W be subspaces of $\mathbb{C}[G]$ satisfying* (V2) *and both containing η_G. Then $(V+W)^\circ = V^\circ + W^\circ$.*

Proof. By Lemma 5.6 we have $V^\circ = V(1-\eta_H)$ and $W^\circ = W(1-\eta_H)$, and since $V+W$ also satisfies (V2) and contains η_G, we also have $(V+W)^\circ = (V+W)(1-\eta_H)$. The lemma follows. □

Recall that (V1) and (V3) are the properties that $Vw \subseteq V$ and $V^\circ w \subseteq V^\circ$, respectively; (V2) has just been used.

Lemma 5.9. *Properties* (V1)*,* (V2) *and* (V3) *are closed under addition of vector subspaces that contain η_G.*

Proof. Let V and W be vector subspaces of $\mathbb{C}[G]$ satisfying (V1)–(V3). Then $(V+W)w = Vw+Ww \subseteq V+W$ giving (V1) and $\pi_{bH}(V+W) = \pi_{bH}(V)+\pi_{bH}(W) \subseteq V+W$, giving (V2). Finally (V3) holds by Lemma 5.8. □

By Corollary 1.6 and ($\star$) in §5.2, if the left-invariant random walk lumps weakly to G/H in the sense of Definition 1.1(b), then L_w (the new name for $V_{\eta_G,w}$) satisfies (V1), (V2), and (V3). We can thus consider the unique maximal subspace of $\mathbb{C}[G]$ satisfying these properties, which we denote J_w.

Lemma 5.10. *Suppose that the irreducible weight w lumps weakly to G/H. The maximal subspace J_w of $\mathbb{C}[G]$ containing η_G and satisfying properties* (V1)*,* (V2)*, and* (V3) *is a left ideal of $\mathbb{C}[G]$. Moreover, $\pi_H(J_w)$ is a left ideal of $\mathbb{C}[H]$ and $J_w = \mathbb{C}[G]\big(\pi_H(J_w)\big)$ is an induced ideal.*

Proof. We check in a very similar way to the proof of Lemma 5.3 that if $g \in G$ then gJ_w satisfies conditions (V1), (V2), and (V3). Then, by maximality of J_w it follows that $gJ_w \subseteq J_w$, and hence J_w is a left ideal. We leave checking (V1) and (V2) to the reader. To check (V3), note that $\ker \Lambda = \mathbb{C}[G](1-\eta_H)$ is a left ideal of $\mathbb{C}[G]$ and so

$$g(V^\circ) = g(V \cap \ker \Lambda) = gV \cap g \ker \Lambda = gV \cap \ker \Lambda = (gV)^\circ.$$

The end is exactly as in the earlier proof: by (V2) we have $\pi_H(J_w) \subseteq J_w$ and so by Proposition 3.17(ii), $J_w = \mathbb{C}[G]\big(\pi_H(J_w)\big)$. □

We call J_w the *maximal Gurvits–Ledoux ideal for w*. As a complex vector space, $J_w = \mathbb{C} \otimes_R V_{\max}(f,P,\alpha)$. We now use J_w to prove Theorem 1.7. See §6 for its algorithmic counterpart.

Theorem 1.7 (Weak lumping test for a distribution)**.** *Let $w \in \mathbb{C}[G]$ be an irreducible weakly lumping weight. For each probability distribution α, the Markov chain* $\mathrm{MC}(\alpha, w)$ *lumps weakly to G/H if and only if $\alpha \in J_w$.*

Proof. Suppose $\mathrm{MC}(\alpha, w)$ lumps weakly to G/H. Consider the minimal space $V_{\alpha,w}$ satisfying (V0), (V1), and (V2). By ($\star$) in §5.2, it also satisfies (V3). By maximality, J_w contains $V_{\alpha,w}$ and so $\alpha \in J_w$. Conversely, suppose that $\alpha \in J_w$. That is, J_w satisfies (V0). By definition, J_w satisfies (V1), (V2), and (V3). Since $J_w = \mathbb{C} \otimes_\mathbb{R} V_{\max}(f,P,\alpha)$, if $\alpha \in J_w$ is a probability vector then $\alpha \in V_{\max}(f,P,\alpha)$ and hence, by Corollary 2.12, $\mathrm{MC}(\alpha, w)$ lumps weakly to G/H. □

5.4. **Gurvits–Ledoux ideals: the general case and the proof of Theorem 1.2.** We have seen that the minimal Gurvits–Ledoux vector space L_w and the maximal Gurvits–Ledoux vector space J_w controlling weak lumping of a weight w to G/H are induced left ideals of $\mathbb{C}[G]$. This motivates the following definition.

Definition 5.11. Let L be a left ideal of $\mathbb{C}[G]$. We say L is a *Gurvits–Ledoux ideal for w* if

(L0) $\eta_G \in L$,
(L1) $Lw \subseteq L$, and
(L2) L is an induced ideal from H to G.

We say that a Gurvits–Ledoux ideal for w is *weakly lumping* if

(L3) $L^\circ w \subseteq L^\circ$,

We denote by $L_{\alpha,w}$ the minimal Gurvits–Ledoux ideal containing the distribution α.

We immediately justify the term 'weakly lumping' in this definition.

Proposition 5.12. *Let L be a Gurvits–Ledoux ideal for the weight w. The left-invariant random walk driven by w lumps weakly to G/H with stable space L if and only if $L^\circ w \subseteq L^\circ$.*

Proof. We use the basic fact proven in Lemma 3.1 that right-multiplication by w is right-multiplication by the transition matrix P of the left-invariant random walk driven by w. Conditions (a), (b), and (c) of Definition 2.10 are satisfied when V is taken to be any Gurvits–Ledoux ideal L. Condition (d) reduces to $L^\circ w \subseteq L^\circ$. □

Corollary 5.13. *If there exists a Gurvits–Ledoux ideal L for the weight w such that $L^\circ w \subseteq L^\circ$, then the left-invariant random walk driven by w lumps weakly to G/H.*

Proof. This follows from Proposition 5.12 and Corollary 2.12. □

It is now natural to ask how L° can be computed. This has an appealing answer in terms of idempotents. Observe that if L is a Gurvits–Ledoux ideal for w then by Proposition 3.17(v) and (L2) there exists an idempotent $e \in \mathbb{C}[H]$ such that $L = \mathbb{C}[G]e$. By (L0), L contains η_G, and so $\eta_G e = \eta_G$. Applying the projection map π_H we obtain $\eta_H e = \eta_H$. Hence $e \in E^\bullet(H)$, the set of idempotents of $\mathbb{C}[H]$ such that $\eta_H e = \eta_H$. Now by Lemma 5.7,

$$L^\circ = L(1-\eta_H) = \mathbb{C}[G]e(1-\eta_H) = \mathbb{C}[G](e - e\eta_H) = \mathbb{C}[G](e-\eta_H).$$

Thus L° is concretely described by idempotent multiplication. Similarly, the following lemma translates conditions (L1) and (L3) into the language of idempotents.

Lemma 5.14. *Let $L = \mathbb{C}[G]e$ be an induced left ideal of $\mathbb{C}[G]$, for some $e \in E^\bullet(H)$. For $w \in \mathbb{C}[G]$ we have*

(i) *$Lw \subseteq L$ if and only if $ew(1-e) = 0$;*
(ii) *$L^\circ w \subseteq L^\circ$ if and only if $(e-\eta_H)w(1-e+\eta_H) = 0$.*

Moreover (i) *and* (ii) *are together equivalent to*

(iii) *$ew(1-e) = 0$ and $(e-\eta_H)w\eta_H = 0$.*

Proof. Observe that if f is an idempotent then $\mathbb{C}[G]f = \{x \in \mathbb{C}[G] : x(1-f) = 0\}$. Applying this with $f = e$ and then $f = e - \eta_H$ (using Lemma 5.7) we get

$$Lw \subseteq L \iff \mathbb{C}[G]ew \subseteq \mathbb{C}[G]e \iff \mathbb{C}[G]ew(1-e) = 0 \iff ew(1-e) = 0$$

proving (i) and very similarly that $L^\circ w \subseteq L^\circ$ if and only if $(e-\eta_H)w(1-e+\eta_H) = 0$ proving (ii). Now multiplying $ew(1-e) = 0$ on the left by $1-\eta_H$ we get $(e-\eta_H)w(1-e) = 0$. Therefore, given that $ew(1-e) = 0$, the conditions $(e-\eta_H)w(1-e+\eta_H) = 0$ and $(e-\eta_H)w\eta_H$ are equivalent. This proves (iii). □

The following further remarks connect Definition 5.11 with the definitions and results presented so far.

(1) If w is an irreducible weight then, as in Lemma 2.13, then it follows from (L1) that $\eta_G \in L_{\alpha,w}$. Thus in the irreducible case (L0) could be replaced by the condition that L contains some probability vector.

(2) By (V2), the subspace $V_{\alpha,w}$ of $\mathbb{C}[G]$ originally defined in §5.1 is closed under the projection maps. Therefore

$$V_{\alpha,w} = \bigoplus_{b\in G/H} \pi_{bH}(V_{w,\alpha}) = \bigoplus_{b\in G/H} b\pi_H(b^{-1}V_{w,\alpha})$$

and it follows that the left ideal of $\mathbb{C}[G]$ generated by $V_{\alpha,w}$ is $\mathbb{C}[G]U$ where $U = \sum_{b\in G/H} \pi_H(b^{-1}V_{w,\alpha})$. It is therefore an induced ideal and so $L_{\alpha,w}$ is simply the smallest ideal containing the subspace $V_{\alpha,w}$.

(3) By Lemma 5.3, the minimal Gurvits–Ledoux space $V_{\eta_G,w}$ is an induced left ideal of $\mathbb{C}[G]$, and so we have

$$V_{\eta_G,w} = L_w = L_{\eta_G,w}$$

where the second equality holds by definition. (Note the first equality is the change in notation introduced after the proof of Lemma 5.7, so our usage of L_w is consistent throughout.)

(4) By Lemma 5.10, the maximal Gurvits–Ledoux space J_w is an induced left ideal and since, by its definition in Lemma 5.10, it satisfies (V3), the left-invariant random walk driven by w lumps stably for the ideal J_w.

We can now prove Theorem 1.2 and Corollary 1.5, which we restate below.

Theorem 1.2. *Let w be an irreducible weight on G and let α be a distribution on G, both thought as elements of $\mathbb{C}[G]$. Then $\mathrm{MC}(\alpha, w)$ lumps weakly to G/H if and only if there exists an idempotent $e \in E^\bullet(H)$ such that*

(i) $\alpha \in \mathbb{C}[G]e$,
(ii) $ew(1-e) = 0$,
(iii) $(e-\eta_H)w\eta_H = 0$.

In this case, for any $t \geq 0$, the conditional distribution of X_t given the sequence of cosets X_0H, ..., $X_{t-1}H$ always belongs to $\mathbb{C}[G]e$.

Proof. Suppose that $\mathrm{MC}(\alpha, w)$ lumps weakly to G/H. Recall that J_w is the maximal Gurvits–Ledoux ideal for w defined in §5.3. By Theorem 1.7, proved at the end of §5.3, we have $\alpha \in J_w$, giving (i). As seen at the start of §3.4, there exists an idempotent $e \in \mathbb{C}[H]$ such that $J_w = \mathbb{C}[G]e$. Since $\eta_G \in J_w$ by (L0), we have $e \in E^\bullet(H)$. By Lemma 5.14 and (L2) and (L3), e satisfies (ii) and (iii).

Conversely, suppose that there is an idempotent $e \in E^\bullet(H)$ satisfying (i), (ii) and (iii). Set $L = \mathbb{C}[G]e$. By (i), L contains α. By Lemma 5.14(ii) and (iii), $Lw \subseteq L$ and $L^\circ w \subseteq L^\circ$. Therefore by Proposition 5.12, $\mathrm{MC}(\alpha, w)$ lumps weakly to G/H with stable ideal L. In particular $\mathrm{MC}(\alpha, w)$ lumps weakly.

The final claim restates the analysis preceding Theorem 2.6. □

Corollary 1.5. *The Markov chain* $\mathrm{MC}(\alpha, w)$ *lumps weakly to left cosets of* G/H *if and only if* $L^\circ_{\alpha,w} w \subseteq L^\circ_{\alpha,w}$.

Proof. Suppose that $\mathrm{MC}(\alpha, w)$ lumps weakly to left cosets of H. Then by Theorem 1.2, there exists an idempotent $e \in E^\bullet(H)$ satisfying conditions (i), (ii), and (iii) from this theorem. By Lemma 5.14 and by minimality, we have $L_{\alpha,w} \subseteq \mathbb{C}[G]e$. In particular, by (iii), we have $(e - \eta_H)w\eta_H = 0$ and so $\mathbb{C}[G]e(1 - \eta_H)w\eta_H = 0$ and thus $L_{\alpha,w}(1 - \eta_H)w\eta_H = 0$. Again by Lemma 5.14, this becomes (L3), as desired.

Conversely suppose that $L^\circ_{\alpha,w} w \subseteq L^\circ_{\alpha,w}$. Then an idempotent generator of $L_{\alpha,w}$ satisfies conditions (i), (ii), and (iii) of Theorem 1.2. □

5.5. **Corollaries for strong and exact lumping.** We summarise the results of this section for these two special cases. Strong lumping is defined in Definition 2.18. Note that conditions (iii)–(vi) in both propositions are independent of α.

Proposition 5.15 (Characterisations of strong lumping)**.** *Let* w *be an irreducible weight. The following are equivalent:*

(i) *the left-invariant random walk driven by* w *lumps strongly to* G/H*;*
(ii) $\mathrm{MC}(\alpha, w)$ *lumps weakly to* G/H *for all initial distributions* α*;*
(iii) $J_w = \mathbb{C}[G]$ *and* $J^\circ_w = \mathbb{C}[G](1 - \eta_H)$*;*
(iv) $\mathbb{C}[G]$ *is a weak lumping Gurvits–Ledoux ideal for* w*;*
(v) $(\ker \Lambda)w \subseteq \ker \Lambda$*;*
(vi) $(1 - \eta_H)w\eta_H = 0$*;*
(vii) *For each* $g \in G$, $w(hgH)$ *is constant for* $h \in H$.

Proof. As we explained after Definition 2.18, strong lumping implies weak lumping starting at an arbitrary initial distribution. Hence (i) implies (ii). In this case, by Theorem 1.7, $J_w = \mathbb{C}[G]$, and then $J^\circ_w = \mathbb{C}[G](1 - \eta_G)$ by Lemma 5.7, giving (iii). It is clear from Definition 5.11 that $\mathbb{C}[G]$ is a Gurvits–Ledoux ideal for any weight. By definition $\mathbb{C}[G]$ is weakly lumping for the weight w if and only if $\mathbb{C}[G]^\circ w \subseteq \mathbb{C}[G]^\circ$; this holds by (V3) for $J_w = \mathbb{C}[G]$. Hence (iii) implies (iv). By definition, $\mathbb{C}[G]^\circ = \ker \Lambda$ so (v) is a restatement of (iv). Now (v) implies (vi) by taking $e = 1$ in Lemma 5.14 and (vi) implies (vii) by Lemma 4.1(ii). Finally suppose that (vii) holds. The Dynkin condition for strong lumping (see Definition 2.18) is that

$$\sum_{x \in bH} w(a^{-1}x) = \sum_{x \in bH} w(a'^{-1}x)$$

for all left cosets bH, all $a, a' \in G$ such that $aH = a'H$, and all $x \in G$. Equivalently, $w(a^{-1}bH) = w((ah)^{-1}bH)$ for all a, $b \in G$ and $h \in H$, and this holds by (vii) since $(ah)^{-1}bH = h^{-1}a^{-1}bH$. Hence (vii) implies (i), completing the cycle. □

Exact lumping is defined in Definition 2.19.

Proposition 5.16 (Exact lumping)**.** *Let w be an irreducible weight. The following are equivalent:*

(i) *the left-invariant random walk driven by w lumps exactly to G/H;*
(ii) $\eta_H w \in \langle b\eta_H : b \in G/H\rangle$*;*
(iii) $L_w = \mathbb{C}[G]\eta_H$ *and* $L_w^\circ = 0$*;*
(iv) $\mathbb{C}[G]\eta_H$ *is a weak lumping Gurvits–Ledoux ideal for w;*
(v) $\mathbb{C}[G]\eta_H w \subseteq \mathbb{C}[G]\eta_H$*;*
(vi) $\eta_H w(1-\eta_H) = 0$*;*
(vii) *For each $g \in G$, $w(Hgh)$ is constant for $h \in H$.*

Moreover if α is an initial distribution then $\mathrm{MC}(\alpha, w)$ *lumps exactly to G/H if and only if one of these conditions holds and, in addition, the restriction of α to each left coset bH is proportional to $b\eta_H$.*

Proof. By the equivalence of (a) and (c) in Lemma 2.20, (i) holds if and only if $\bigoplus_{b\in G/H}\langle b\eta_H\rangle$ is preserved by right multiplication by w. Thus (i) and (ii) are equivalent. Moreover, by Corollary 2.21, $\mathrm{MC}(\alpha, w)$ lumps exactly if and only if (i) holds and the restriction of α to each left coset bH is proportional to $b\eta_H$. To complete the proof it suffices to prove that conditions (ii)–(vii) are equivalent.

If (ii) holds then, considering the definition of $V_{\eta_G,w}$ in §5.1, we have

$$\bigoplus_{b\in G/H} \langle b\eta_H\rangle = V_{\eta_G,w}.$$

This space is by definition (see the change of notation after Lemma 5.7), the minimal Gurvits–Ledoux ideal L_w. Therefore (ii) implies that $L_w = \mathbb{C}[G]\eta_H$. In this case, by Lemma 5.14, $L_w^\circ = \mathbb{C}[G]\eta_H(1-\eta_H) = 0$. Hence (ii) implies (iii). Suppose that (iii) holds. Then $L_w = \mathbb{C}[G]\eta_H$ satisfies $L_w w \subseteq L_w$ by (V1) and clearly $\eta_G \in L_w$ and since $\eta_H \in E(H)$, Proposition 3.17(v) implies that L_w is a Gurvits–Ledoux ideal in the sense of Definition 5.11. By (iii) we have $L_w^\circ = 0$, hence $L_w^\circ w \subseteq L_w^\circ$ and we have (iv). Part (v) simply restates that $\mathbb{C}[G]\eta_H w \subseteq \mathbb{C}[G]\eta_H$, so (iv) implies (v), and since $\eta_H w \in \mathbb{C}[G]\eta_H$ if and only if $\eta_H w(1-\eta_H) = 0$, (v) implies (vi). Since (vi) is equivalent to $\eta_H w = \eta_H w \eta_H$, Lemma 4.1(i) and (iii) applied with $T = H$ imply that $\eta_H w$ is constant on each right coset Hg in a given double coset, hence (vi) and (vii) are equivalent. Finally (vi) implies (ii) since $\langle b\eta_H : b \in G/H\rangle$ is the kernel of right multiplication by $1-\eta_H$. □

We remark that the equivalence of (i) and (vii) in Proposition 5.15 and Proposition 5.16 proves Corollary 1.10. We later deduce this result in a more conceptual way as a corollary of Theorem 1.9: see §10.

We end this section with a joint corollary of Theorem 1.2 and the two propositions above that deals with cases when H is very small.

Corollary 5.17. *Let H be a subgroup of G, and suppose that $|H| \le 3$. Let w be an irreducible weight and α any probability distribution on G. If* $\mathrm{MC}(\alpha, w)$ *lumps weakly then it lumps either strongly or exactly.*

Proof. If $|H| = 1$ then every weight lumps both strongly and exactly to $G/H = G$. The only possibility in Theorem 1.2 is $e = \eta_H = \mathrm{id}_H$, and conditions (vi) in Proposition 5.15 and Proposition 5.16 hold trivially.

Now suppose $|H| = 2$. Let $H = \langle h \rangle$. Then $E^\bullet(H) = \{\mathrm{id}_H, \eta_H\}$. There are two possible cases in Theorem 1.2. If $e = \eta_H$, then the condition $ew(1-e) = 0$ gives condition (vi) of Proposition 5.16 so we have exact lumping. If instead $e = \mathrm{id}_H$ the condition $(e - \eta_H)w\eta_H = 0$ gives condition (vi) of Proposition 5.15 so we have strong lumping.

Finally, suppose $|H| = 3$. Again let $H = \langle h \rangle$. There are three primitive idempotents

$$\begin{aligned}\eta_H &= \tfrac{1}{3}(1 + h + h^2),\\ \xi_H &= \tfrac{1}{3}(1 + \zeta^2 h + \zeta h^2),\\ \overline{\xi}_H &= \tfrac{1}{3}(1 + \zeta h + \zeta^2 h^2)\end{aligned}$$

where ζ is a complex primitive third root of unity. Thus $\zeta^2 = \overline{\zeta}$. Since the initial distribution α and the weight w take real values, the minimal Gurvits–Ledoux ideal $L_{\alpha,w}$ defined in Definition 5.11 is closed under complex conjugation. If $L_{\alpha,w} = \mathbb{C}[G]\eta_H$ then by Proposition 5.16(iv), $\mathrm{MC}(\alpha, w)$ lumps exactly to G/H. Otherwise $L_{\alpha,w}$ cannot be either of $\mathbb{C}[G](\eta_H + \xi_H)$ or $\mathbb{C}[G](\eta_H + \overline{\xi}_H)$ since these are exchanged by complex conjugation. The only remaining possibility is that $L_{\alpha,w} = \mathbb{C}[G]$, and then Proposition 5.15(iv) implies that $\mathrm{MC}(\alpha, w)$ lumps strongly to G/H. □

6. Tests for weak lumping

Corollary 1.6 and Theorem 1.7 are characterisations of weak lumpability of a weight w and weak lumping of $\mathrm{MC}(\alpha, w)$ for a distribution α. They rely on the computation of the left ideals L_w and J_w given a weight w. In this section, we provide two practical computational procedures to compute these left ideals of $\mathbb{C}[G]$. Each algorithm uses a nested sequence of left ideals of $\mathbb{C}[H]$ that eventually stabilises, in one case to $\pi_H(L_w)$ and in the other to $\pi_H(J_w)$, respectively.

By Lemmas 5.3 and 5.10, L_w (which was earlier denoted $V_{\alpha_G,w}$) and J_w are induced ideals of $\mathbb{C}[G]$, and so satisfy $L_w = \mathbb{C}[G]\big(\pi_H(L_w)\big)$ and $J_w = \mathbb{C}[G]\big(\pi_H(J_w)\big)$ by Proposition 3.17(ii). When the order of H is small compared to that of G, computing ideals of $\mathbb{C}[H]$ and hence induced ideals of $\mathbb{C}[G]$ is significantly more efficient than computing vector subspaces of $\mathbb{C}[G]$; it is in this sense that our algorithms become more powerful than those of [31] and [20], for instance Corollary 2.9 which gave an algorithm for determining whether a general DTHMC $\mathrm{MC}(\alpha, P)$ lumps weakly under $f : A \to B$. Magma [6] code that implements the two algorithms for calculating L_w and J_w is available as part of the arXiv version of this paper.

6.1. **Weak lumping test for a weight.** The characterisation of weak lumpability provided by Corollary 1.6 relies on L_w, the minimal Gurvits–Ledoux ideal for w. We construct it algorithmically as follows.

Start with the left ideal $M_0 = \mathbb{C}[H]\eta_H$. Define inductively

$$M_n := \pi_H\left(\mathbb{C}[G]M_{n-1} + \mathbb{C}[G]\bigoplus_{bH\in G/H}\pi_{bH}(M_{n-1}w)\right).$$

Since L_w is a left ideal and satisfies (L1) and (L2), we have $\mathbb{C}[G]M_n \subseteq L_w$. By construction $M_n \supseteq M_{n-1}$ so the sequence $(\dim(M_n))_{\geq 0}$ is increasing in n, takes integer values, and is bounded above by $\dim \mathbb{C}[G]$. So it must stabilise: there exists N such that $M_n = M_N$ for all $n \geq N$. In each step, $\mathbb{C}[G]M_n$ is a left ideal of $\mathbb{C}[G]$ containing η_G, and thus $\mathbb{C}[G]M_N$ is too. This shows that $\mathbb{C}[G]M_N$ satisfies (L0). Since M_N is by definition an ideal of $\mathbb{C}[H]$, the ideal $\mathbb{C}[G]M_N$ is an induced ideal, as required by (L2). Since $\mathbb{C}[G]M_N = \mathbb{C}[G]\pi_H(M_N)$, we have

$$\mathbb{C}[G] \bigoplus_{bH \in G/H} \pi_{bH}(M_N w) \subseteq \mathbb{C}[G]M_N.$$

This gives (L1). Since $\mathbb{C}[G]M_n$ is contained in L_w, we obtain $L_w = \mathbb{C}[G]M_N$ by minimality.

Example 6.1. We take $G = \mathrm{Sym}_4$ and $H = \mathrm{Sym}_{\{2,3,4\}}$. In §1.2 we showed that the weight $w = (1-\lambda)\mathrm{id} + \frac{\lambda}{3}\big((1,4)(2,3) + (1,4,3) + (1,4,2,3)\big)$ defined in (1.3) lumps weakly to G/H with stable ideal $\mathbb{C}[G]\eta_T$, where $T = \mathrm{Sym}_{\{2,3\}}$. We now use Corollary 1.6 to find the left ideal L_w when $0 < \lambda < 1$ and deduce that w does not lump stably for any proper subideal of $\mathbb{C}[G]\eta_T$. Following the construction above, we set $M_0 = \mathbb{C}[H]\eta_H = \langle \eta_H \rangle$. Calculation shows that the normalized projections to the left cosets H, $(1,2)H$, $(1,3)H$ and $(1,4)H$ of $\eta_H w$ are

$$\begin{aligned}
\pi_H(\eta_H w) &= \eta_H \\
\pi_{(1,2)H}(\eta_H w) &= \tfrac{1}{2}(1,4,2) + \tfrac{1}{2}(1,4,3,2) \\
\pi_{(1,3)H}(\eta_H w) &= \tfrac{1}{2}(1,4,3) + \tfrac{1}{2}(1,4,2,3) \\
\pi_{(1,4)H}(\eta_H w) &= \tfrac{1}{2}(1,4) + \tfrac{1}{2}(1,4)(2,3)
\end{aligned}$$

and the ideal of $\mathbb{C}[G]$ generated by the projections is $\mathbb{C}[G]\eta_T$. Therefore $M_1 = \mathbb{C}[H]\eta_T$. Since we know that w lumps weakly to G/H with stable ideal $\mathbb{C}[G]\eta_T$, it follows by minimality of L_w that $L_w = \mathbb{C}[G]\eta_T$. Alternatively this can be checked by calculating directly that $M_3 = M_2$. Hence the minimal Gurvits–Ledoux space L_w is $\mathbb{C}[\mathrm{Sym}_4]\eta_T$. A very similar calculation shows that if $w' = \eta_T w$ as earlier then $L_{w'} = \mathbb{C}[\mathrm{Sym}_4]\eta_T$, and so the stable lumping ideals found in the earlier example were minimal in both cases.

6.2. **Weak lumping test for a distribution.** Let $w \in \mathbb{C}[G]$ be an irreducible weight, and assume that it lumps weakly on left cosets of H. The set of distributions α such that $\mathrm{MC}(\alpha, w)$ lumps weakly on left cosets of H is J_w by Theorem 1.7. In this section, we provide a practical computational procedure to compute J_w. It may be compared with the general algorithm that we gave after Corollary 2.16 for computing the maximal Gurvits–Ledoux space $V_{\max}(f, P)$ associated to an irreducible stochastic matrix that lumps weakly under $f : A \to B$.

Let $A_0 = \mathbb{C}[H]$. We have $J_w \subseteq \mathbb{C}[G]A_0$. Of the conditions defined at the start of §5.1, the left ideal $\mathbb{C}[G]A_0$ satisfies conditions (V1) and (V2), but not necessarily condition (V3). Also, note $L_w \subseteq \mathbb{C}[G]A_0$. Define inductively B_n so that B_n° is the largest subspace of A_n° such that

$$\mathbb{C}[G]B_n^\circ w \subseteq \mathbb{C}[G]A_n^\circ.$$

The largest such subspace is well defined, since this property is closed under sum. It is a left ideal of $\mathbb{C}[H]$ since this property is closed under left $\mathbb{C}[H]$-multiplication. Moreover, $L_w^\circ w \subseteq L_w^\circ \subseteq \mathbb{C}[G]A_n^\circ$ and thus L_w is a subideal of $\mathbb{C}[G]B_n$. Define A_{n+1} so that A_{n+1}° is the largest subspace of B_n° such that

$$\big(\mathbb{C}[G]A_{n+1}^\circ \oplus \mathbb{C}[G]\eta_H\big)w \subseteq \mathbb{C}[G]B_n.$$

Since $\mathbb{C}[G]\eta_H w \subseteq L_w w \subseteq L_w \subseteq \mathbb{C}[G]B_n$, the largest such space is well defined, and it is a left ideal of $\mathbb{C}[H]$. We therefore have a sequence of nested left ideals of $\mathbb{C}[G]$ given by

$$\mathbb{C}[G]A_0 \supseteq \mathbb{C}[G]B_0 \supseteq \mathbb{C}[G]A_1 \supseteq \mathbb{C}[G]B_1 \supseteq \cdots, \tag{6.1}$$

which is bounded below by J_w, since J_w is defined to be the largest space satisfiying (V1)–(V3). Therefore, the sequence stabilises: there exists N such that $A_n = B_n = A_N$ for all $n \geq N$. Moreover, every term in (6.1) is an induced ideal, and hence so is A_N. This shows that $\mathbb{C}[G]A_N$ satisfies (V2). By construction,

$$\mathbb{C}[G]B_N^\circ w \subseteq \mathbb{C}[G]A_N^\circ \quad \text{and} \quad \mathbb{C}[G]A_N w \subseteq \mathbb{C}[G]B_N,$$

which implies that $\mathbb{C}[G]A_N = \mathbb{C}[G]B_N$ satisfies (V1) and (V3). We conclude that $\mathbb{C}[G]A_N = J_w$ by maximality of J_w.

Example 6.2. Again we use the example from §1.2, taking $G = \mathrm{Sym}_4$ and $H = \mathrm{Sym}_{\{2,3,4\}}$ and the weight w, now in the uniform case with $\lambda = \frac{3}{4}$, so $w = \frac{1}{4}\big(\mathrm{id} + (1,4)(2,3) + (1,4,3) + (1,4,2,3)\big)$. Following the construction above we take $A_0 = \mathbb{Q}[H]$ and find using computer algebra that B_0° is the 3-dimensional left ideal of $\mathbb{C}[H]$ generated by $1 - (2,4) - (3,4) + \eta_T$, where $T = \langle (2,3) \rangle \leq H$. Noting that $\eta_H(1 - (2,4) - (3,4)) = -\eta_H$, it follows that B_0 is the 4-dimensional left ideal of $\mathbb{C}[H]$ generated by $1 - (2,4) - (3,4)$. A similar computer algebra calculation now show that A_1° is 2-dimensional, spanned by

$$\mathrm{id}_{\mathrm{Sym}_4} - (2,4,3) + (2,3) - (2,4) = 2\,\mathrm{id}_{\mathrm{Sym}_4} + (2,3,4) + (3,4) + 2(2,3) - 6\eta_H,$$
$$(2,3,4) - (2,4,3) - (2,4) + (3,4) = \mathrm{id}_{\mathrm{Sym}_4} + 2(2,3,4) + 2(3,4) + (2,3) - 6\eta_H.$$

It follows that $1 + (2,3) \in A_1$, and so A_1 is the 3-dimensional ideal $\mathbb{C}[H]\eta_T$. We know that w lumps weakly to G/H with this stable ideal, so the algorithm stabilises at this point: $B_1^\circ = A_1^\circ$, $B_1 = A_1$, $A_2^\circ = A_1^\circ$ and $A_2 = A_1$. (Again this may be verified by computer algebra.) We conclude that $J_w = \mathbb{C}[G]\eta_T$ and, given the previous example, that in this case the minimal and maximal Gurvits–Ledoux ideals L_w and J_w coincide. If we instead take the strongly lumping weight $w' = \eta_T w$ then calculation shows that $B_0^\circ = \mathbb{C}[H](1 - \eta_H)$ and $B_0 = \mathbb{C}[H]$, and now it is immediate that $A_1^\circ = \mathbb{C}[H](1 - \eta_H)$ and $A_1 = \mathbb{C}[H]$. Hence the algorithm stabilises one step sooner and we obtain $J_w = \mathbb{C}[G]$, as expected from Proposition 5.15(iii).

7. The structure of the set of all weakly lumping weights

By Definition 1.1, a weight w lumps weakly to G/H if $\mathrm{MC}(\alpha, w)$ lumps weakly to G/H for some initial distribution α. In (1.1) we defined the sets

$$\Theta(e) = \big\{ w \in \mathbb{C}[G] : ew(1-e) = 0, (e - \eta_H)w\eta_H = 0 \big\}.$$

As we mentioned after Theorem 1.2, it follows easily from this theorem that the set of weakly lumping weights is $\Delta \cap \Theta$ where

$$\Theta = \bigcup_{e \in E^{\bullet}(H)} \Theta(e). \tag{7.1}$$

and $\Delta \subseteq \mathbb{R}[G]$ is the simplex of probability distributions. In this section we begin by studying the sets $\Theta(e)$, which turn out to have remarkable algebraic properties. In particular, they are subalgebras of $\mathbb{C}[G]$. We compute the dimension of each $\Theta(e)$ in Corollary 7.3. We continue by considering when the union in (7.1) is redundant, in the sense that one of these sets is contained in another. We then prove Proposition 1.8.

7.1. **Weak lumping algebras.** Let $e \in E^{\bullet}(H)$ be an idempotent, and let $L = \mathbb{C}[G]e$. Recall that $\Theta(e)$ is the set of weights w for which $L = \mathbb{C}[G]e$ is a weakly lumping Gurvits–Ledoux ideal in the sense of Definition 5.11. By Lemma 5.14(i) and (ii),

$$\begin{aligned}\Theta(e) &= \{w \in \mathbb{C}[G] : Lw \subseteq L,\ L^{\circ}w \subseteq L^{\circ}\} \\ &= \{w \in \mathbb{C}[G] : ew(1-e) = 0,\ (e - \eta_H)w(1 - e + \eta_H) = 0\}.\end{aligned}$$

The set $\{w \in \mathbb{C}[G] : Lw \subseteq L\}$ is the right idealizer $\mathrm{RId}_{\mathbb{C}[G]}(L)$ of L. By Lemma 3.24,

$$\Theta(e) = \mathrm{RId}_{\mathbb{C}[G]}(L) \cap \mathrm{RId}_{\mathbb{C}[G]}(L^{\circ}), \tag{7.2}$$

is a subalgebra of $\mathbb{C}[G]$. We call it the *weak lumping algebra of e*.

Lemma 7.1. *Let $e \in E^{\bullet}(H)$ be an idempotent. The weak lumping algebra of e is a parabolic subalgebra of $\mathbb{C}[G]$. It satisfies*

$$\Theta(e) = (\mathbb{C}[G](e - \eta_H) + (1-e)\mathbb{C}[G]) \oplus \eta_H \mathbb{C}[G] \eta_H.$$

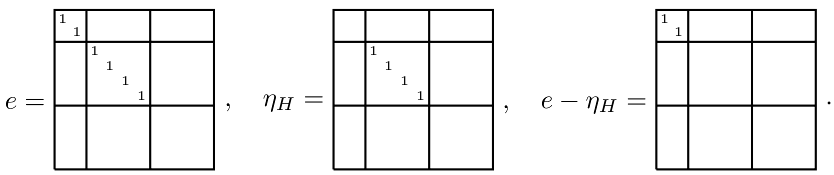
Proof. Let $L = \mathbb{C}[G]e$ and consider $\Theta(e) = \mathrm{RId}_{\mathbb{C}[G]}(L) \cap \mathrm{RId}_{\mathbb{C}[G]}(L^{\circ})$. Since $e \in E^{\bullet}(H)$ there is an idempotent decomposition of the identity element of $\mathbb{C}[G]$:

$$1 = (e - \eta_H) + \eta_H + (1 - e).$$

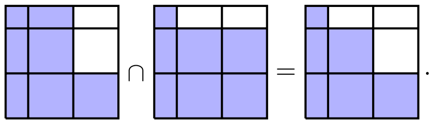
Using Proposition 3.6 we may choose a Wedderburn isomorphism such that, on the block $\mathrm{Mat}(V)$ corresponding to the irreducible character $\chi_V \in \mathrm{Irr}\, G$, the elements e and η_H are sent to the diagonal matrices

$$e = \mathrm{diag}(\underbrace{1, ..., 1, 1, ..., 1}_{\langle \chi_L, \chi_V \rangle}, 0, ..., 0) \text{ and } \eta_H = \mathrm{diag}(\underbrace{0, ..., 0}_{\langle \chi_{L^{\circ}}, \chi_V \rangle}, \underbrace{1, ..., 1}_{\langle \mathbb{1}\uparrow_H^G, \chi_V \rangle}, 0, ..., 0).$$

We represent this diagrammatically by



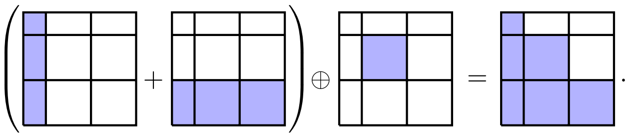
By Lemma 3.24, both $\mathrm{RId}_{\mathbb{C}[G]}(L)$ and $\mathrm{RId}_{\mathbb{C}[G]}(L^{\circ})$ are standard parabolics, and their intersection is again a standard parabolic: using (7.2) the part of this intersection in the block $\mathrm{Mat}(V)$ is

$\Theta(e) \cap \mathrm{Mat}(V) =$ 

The part of the right-hand side in the lemma in the block $\mathrm{Mat}(V)$ is

$$\big((\mathbb{C}[G](e-\eta_H) + (1-e)\mathbb{C}[G]) \oplus \eta_H\mathbb{C}[G]\eta_H\big) \cap \mathrm{Mat}(V),$$

which diagrammatically becomes



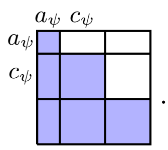
Since the diagrams agree, the lemma holds for the part of the weak lumping algebra $\Theta(e)$ in the block $\mathrm{Mat}(V)$. The lemma follows by summing over all blocks in the Wedderburn decomposition. □

Remark 7.2. A consequence of the formula of Lemma 7.1 is that the subalgebra $\eta_H\mathbb{C}[G]\eta_H$, shown diagrammatically by the second summand above, is a direct summand common to all weak lumping algebras. This subalgebra can be identified with the set of H-bi-invariant functions on $\mathbb{C}[G]$; this is the Hecke algebra seen in Theorem 1.11 and its proof in §9.2.

Corollary 7.3. *Let $e \in E^\bullet(H)$ be an idempotent and let $L = \mathbb{C}[G]e$. For each irreducible character $\psi \in \mathrm{Irr}\, G$, define $a_\psi = \langle \chi_{L^\circ}, \psi\rangle$, $c_\psi = \langle \mathbb{1}_H\!\uparrow_H^G, \psi\rangle$, and $d_\psi = \dim \psi = \psi(1)$. Then,*

$$\dim \Theta(e) = \sum_{\psi \in \mathrm{Irr}\, G} (a_\psi^2 + a_\psi c_\psi + c_\psi^2 - a_\psi d_\psi - c_\psi d_\psi + d_\psi^2).$$

Proof. Choose a Wedderburn isomorphism as in the proof of Theorem 7.1. On each block $\mathrm{Mat}(V)$ corresponding to the character $\psi \in \mathrm{Irr}\, G$, the intersection $\Theta(e) \cap \mathrm{Mat}(V)$ is mapped to the set of $d_\psi \times d_\psi$ matrices of the form



The dimension as a vector space is therefore

$$a_\psi^2 + (a_\psi + c_\psi)c_\psi + d_\psi(d_\psi - a_\psi - c_\psi).$$ □

Example 7.4 (Exact lumping)**.** By Proposition 5.16, the Markov chain driven by w lumps exactly if and only if $\mathbb{C}[G]\eta_H$ is a weakly lumping Gurvits–Ledoux ideal for w. Equivalently, if and only if $w \in \Theta(\eta_H)$. Then, $\chi_L = \mathbb{1}\!\uparrow_H^G$ is the induced trivial character of H. We have $a_\psi = 0$ for all $\psi \in \mathrm{Irr}\, G$. Therefore,

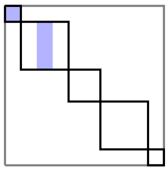
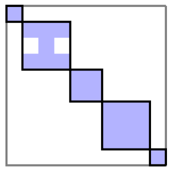
$$\dim \Theta(\eta_H) = \sum_{\psi \in \mathrm{Irr}\, G} \big(c_\psi^2 + d_\psi(d_\psi - c_\psi)\big).$$

In terms of characters, letting ϕ_G denote the regular character of G, we can rewrite the above as

$$\dim \Theta(\eta_H) = \langle \mathbb{1}\!\uparrow_H^G, \mathbb{1}\!\uparrow_H^G \rangle + \langle \phi_G, \phi_G - \mathbb{1}\!\uparrow_H^G \rangle = |H \backslash G / H| + |G| - \frac{|G|}{|H|},$$

where we used Lemma 4.2 (Mackey's rule for the trivial character) to compute that $\langle \mathbb{1}\!\uparrow_H^G, \mathbb{1}\!\uparrow_H^G \rangle = |H \backslash G / H|$. Corollary 1.10 (proved in §10 below) gives another method for computing this dimension.

Example 7.5. Let $H = \mathrm{Sym}_3$ and $G = \mathrm{Sym}_4$. Refer to Examples 3.7 and 3.8 for description of their respective irreducible representations and Wedderburn decompositions. We choose the Wedderburn isomorphism of $\mathbb{C}[G]$ as in Example 3.19 so, in particular, the partitions labelling the blocks from top-left to bottom-right are 4, 31, 22, 211, 1111. We saw that in this chosen isomorphism, $L = S^3\!\uparrow_H^G$ corresponds to the submodule shown left below, and so $\mathrm{RId}(L)$ is as drawn right below:

 ,  .

Since in this case $L^\circ = \varnothing$, we have $\mathrm{RId}(L) = \Theta(\eta_H)$. The weak lumping algebra $\Theta(\eta_H)$ is therefore of dimension $22 = 2 + 24 - 4$, as given by the formula of Example 7.4. Note that this weak lumping algebra is a product of parabolic subalgebras, but, under this Wedderburn isomorphism, the parabolic subalgebra for the block corresponding to the irreducible S^{31} is *not* a standard parabolic.

7.2. **Containment of weak lumping algebras.** Whenever H is nonabelian, the union (7.1) defining Θ is over an uncountable set. We want to understand to what degree this expression is redundant. That is, for which idempotents e, $\tilde{e} \in E^\bullet(H)$ do we have $\Theta(e) \subseteq \Theta(\tilde{e})$?

The following definition defines Borel and parabolic subalgebras of the group algebra $\mathbb{C}[G]$.

Definition 7.6. Let $\mathbb{C}[G] \cong \bigoplus_{V \in \mathrm{Irr}(G)} \mathrm{Mat}(V)$ be a fixed Wedderburn isomorphism of $\mathbb{C}[G]$. Fix an isomorphism of each $\mathrm{Mat}(V)$ with the matrix algebra $\mathrm{Mat}_{\dim V}(\mathbb{C})$. Under these isomorphisms:

(a) The *standard Borel* subalgebra of $\mathbb{C}[G]$ is the product of the standard Borel subalgebras of lower triangular matrices in each factor $\mathrm{Mat}(V)$;

(b) A subalgebra of $\mathbb{C}[G]$ is *Borel* if it is conjugate by an element of $\mathbb{C}[G]^\times$ to the standard Borel;

(c) A subalgebra of $\mathbb{C}[G]$ is *parabolic* if it is conjugate by an element of $\mathbb{C}[G]^\times$ to a subalgebra containing the standard Borel.

Recall that χ_V denotes the character of a representation V.

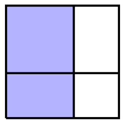
Theorem 7.7. *Let e, $\tilde{e} \in E^\bullet(H)$ be idempotents. Set $L = \mathbb{C}[G]e$ and $\tilde{L} = \mathbb{C}[G]\tilde{e}$. We have $\Theta(e) \subseteq \Theta(\tilde{e})$ if and only if for each irreducible representation $V \in \mathrm{Irr}(G)$ the following two conditions hold:*

(i) *if $\langle \mathbb{1}\!\uparrow_H^G, \chi_V \rangle \neq 0$ then $\tilde{L}^\circ \cap \mathrm{Mat}(V) = L^\circ \cap \mathrm{Mat}(V)$;*

(ii) $\tilde{L}^\circ \cap \mathrm{Mat}(V)$ *is either* 0, *or* $L^\circ \cap \mathrm{Mat}(V)$, *or* $\mathrm{Mat}(V)$.

The proof of this theorem will follow from four lemmas. The first two are elementary results from geometric representation theory; we include proofs for completeness' sake.

We say a left ideal L of $\mathrm{Mat}_d(\mathbb{C})$ is an *initial column span* if it is of the form $\mathrm{Mat}_d(\mathbb{C})e$ for $e = \mathrm{diag}(1, ..., 1, 0, ..., 0)$. That is, if we can write

$L =$  .

Let B_d be the standard Borel subalgebra of lower triangular matrices in $\mathrm{Mat}_d(\mathbb{C})$.

Lemma 7.8. *If the right idealizer* $\mathrm{RId}_{\mathrm{Mat}_d(\mathbb{C})}(L)$ *of a left ideal* L *of* $\mathrm{Mat}_d(\mathbb{C})$ *contains* B_d *then* L *is an initial column span.*

Proof. Let $L = \mathrm{Mat}_d(\mathbb{C})e$ for an idempotent e. By hypothesis, we have $B_d \subseteq \mathrm{RId}_{\mathrm{Mat}_d(\mathbb{C})}(L)$. Since e is an idempotent, we can write

$$e = y^{-1} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix} y$$

for some invertible $y \in \mathrm{Mat}_d(\mathbb{C})$. We deduce yLy^{-1} is an initial column span and that $\mathrm{RId}_{\mathrm{Mat}_d(\mathbb{C})}(yLy^{-1}) = y\,\mathrm{RId}_{\mathrm{Mat}_d(\mathbb{C})}(L)y^{-1}$ is a standard parabolic as in the proof of Lemma 3.24. But by Proposition 3.23, two standard parabolics are conjugate if and only if they are equal. Thus y normalises $\mathrm{RId}_{\mathrm{Mat}_d(\mathbb{C})}(L)$. Reasoning as in Remark 3.25 and using Lemma 3.21 we have

$$\begin{aligned} y &\in N_{\mathrm{Mat}_d(\mathbb{C})^\times}\big(\mathrm{RId}_{\mathrm{Mat}_d(\mathbb{C})}(L)\big) \\ &= \mathrm{RId}_{\mathrm{Mat}_d(\mathbb{C})}(L) \cap \mathrm{Mat}_d(\mathbb{C})^\times \\ &= N_{\mathrm{Mat}_d(\mathbb{C})^\times}(L). \end{aligned}$$

Hence $yLy^{-1} = L$ is an initial column span. □

The second lemma is a combinatorial exercise. It is most natural after the following description of standard parabolics. The symmetric group Sym_d is generated by the simple transpositions $s_i = (i, i{+}1)$ for $i \in \{1, \dots, d{-}1\}$. We define the *parabolic subgroup* of Sym_d indexed by a tuple $\mathbf{a} = (a_1, \dots, a_k)$, where $0 \le a_1 < \dots \le a_k \le d$, to be the subgroup of Sym_n generated by the s_i for $i \notin \{a_1, \dots, a_k\}$, and denote it by $\mathrm{Sym}_d(\mathbf{a})$. Identifying Sym_d with the group of permutation matrices, we can write the standard parabolic subalgebras of $\mathrm{Mat}_d(\mathbb{C})$ as $P_d(\mathbf{a}) = B_d\,\mathrm{Sym}_d(\mathbf{a})B_d$.

Lemma 7.9. *Fix* $0 \le c \le d$ *and two parameters* $0 \le a, \tilde{a} \le d - c$. *Let* $b = a + c$ *and* $\tilde{b} = \tilde{a} + c$. *Then,* $P_d(a, b) \subseteq P_d(\tilde{a}, \tilde{b})$ *if and only if*

(i) $\tilde{a} = 0$ *and* $c \in \{0, a, d\}$,
(ii) $\tilde{a} = a$,
(iii) $\tilde{a} = a + c$ *and* $2c = d - a$, *or*
(iv) $\tilde{a} = d$ *and* $c = 0$.

Proof. We have the containment $P_d(a,b) \subseteq P_d(\tilde{a},\tilde{b})$ if and only if $\{0,a,b,d\} \supseteq \{0,\tilde{a},\tilde{b},d\}$, and so if and only if $\{0,a,a+c,d\} \supseteq \{0,\tilde{a},\tilde{a}+c,d\}$.

- If $\tilde{a}=0$, then $\{0,a,a+c,d\} \supseteq \{0,c,d\}$ gives $c \in \{0,a,d\}$.
- If $\tilde{a}=a$, then the containment evidently holds.
- If $\tilde{a}=a+c$, then $\{0,a,a+c,d\} \supseteq \{0,a+c,a+2c,d\}$ gives either $c=0$ (which gives $\tilde{a}=a$, as above) or $a+2c=d$.
- If $\tilde{a}=d$, then $c=0$. □

Lemma 7.10. *Let e, $\tilde{e} \in E^\bullet(H)$ be idempotents. Set $L = \mathbb{C}[G]e$ and $\tilde{L} = \mathbb{C}[G]\tilde{e}$. The parabolic subalgebras $\Theta(e)$ and $\Theta(\tilde{e})$ of $\mathbb{C}[G]$ share a common Borel if and only if for each irreducible representation $V \in \operatorname{Irr}(G)$ the following two conditions hold:*

- *if $\langle \mathbb{1}_H \uparrow_H^G, \chi_V \rangle \neq 0$ then $L^\circ \cap \operatorname{Mat}(V) = \tilde{L}^\circ \cap \operatorname{Mat}(V)$;*
- *either $L^\circ \cap \operatorname{Mat}(V) \subseteq \tilde{L}^\circ \cap \operatorname{Mat}(V)$ or $\tilde{L}^\circ \cap \operatorname{Mat}(V) \subseteq L^\circ \cap \operatorname{Mat}(V)$.*

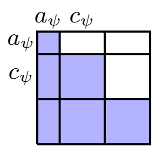
Proof. Given an irreducible module $V \in \operatorname{Irr}(G)$ and its character $\psi \in \operatorname{Irr} G$, define $a_\psi := \langle \chi_{L^\circ}, \psi \rangle$ as in Corollary 7.3, and similarly let $\tilde{a}_\psi = \langle \chi_{\tilde{L}^\circ}, \psi \rangle$.

Suppose $\Theta(e)$ and $\Theta(\tilde{e})$ share a common Borel subalgebra. Fix a Wedderburn isomorphism sending this Borel to the standard Borel of $\bigoplus_V \operatorname{Mat}(V)$, in the sense of Definition 7.6. Then, by Lemma 7.8, each of L, L°, $\tilde{L}$, and $\tilde{L}^\circ$ are sent to initial column spans on each Wedderburn block. In each Wedderburn block $\operatorname{Mat}(V)$, we have

$$\begin{cases} L^\circ \cap \operatorname{Mat}(V) \subseteq \tilde{L}^\circ \cap \operatorname{Mat}(V) & \text{if } a_\psi \leq \tilde{a}_\psi, \\ \tilde{L}^\circ \cap \operatorname{Mat}(V) \subseteq L^\circ \cap \operatorname{Mat}(V) & \text{if } a_\psi \geq \tilde{a}_\psi \end{cases}$$

satisfying the second condition.

Fix $V \in \operatorname{Irr}(G)$ with irreducible character ψ. Suppose that $\langle \mathbb{1}_H \uparrow_H^G, \psi_V \rangle \neq 0$. Since $L = L^\circ \oplus \mathbb{C}[G]\eta_H$, we can recover $\mathbb{C}[G]\eta_H \cap \operatorname{Mat}(V)$ as the span of the columns of $L \cap \operatorname{Mat}(V)$ which are not in $L^\circ \cap \operatorname{Mat}(V)$. (The relevant columns are the first a_ψ columns in the diagram below, repeated from Corollary 7.3.)



Similarly, we can recover $\mathbb{C}[G]\eta_H \cap \operatorname{Mat}(V)$ as the span of those columns of $\tilde{L} \cap \operatorname{Mat}(V)$ which are not in $\tilde{L}^\circ \cap \operatorname{Mat}(V)$. We deduce $a_\psi = \tilde{a}_\psi$ and therefore $L^\circ \cap \operatorname{Mat}(V) = \tilde{L}^\circ \cap \operatorname{Mat}(V)$ as required by the first condition.

Conversely, suppose both conditions are satisfied. Then there is a Wedderburn decomposition sending L, L°, $\tilde{L}$, and $\tilde{L}^\circ$ to initial column spans on each Wedderburn component. Under this Wedderburn isomorphism $\Theta(e)$ and $\Theta(\tilde{e})$ are standard, and they share the standard Borel subalgebra B_d. □

Lemma 7.11. *Let e, $\tilde{e} \in E^\bullet(H)$ be idempotents. Set $L = \mathbb{C}[G]e$ and $\tilde{L} = \mathbb{C}[G]\tilde{e}$. We have $\Theta(e) \subseteq \Theta(\tilde{e})$ if and only if*

- *$\Theta(e)$ and $\Theta(\tilde{e})$ share a common Borel, and*
- *$\tilde{L}^\circ \cap \operatorname{Mat}(V)$ is either 0, or $L^\circ \cap \operatorname{Mat}(V)$, or $\operatorname{Mat}(V)$.*

Proof. We use the notation a_ψ, c_ψ, d_ψ as in Corollary 7.3, and $\tilde{a}_\psi$ as above. Suppose that $\Theta(e) \subseteq \Theta(\tilde{e})$. In particular, both parabolic algebras share a common Borel subalgebra. Fix a Wedderburn isomorphism sending this Borel to the standard Borel of $\bigoplus_V \mathrm{Mat}(V)$. Then L, L°, $\tilde{L}$, and $\tilde{L}^\circ$ are sent to initial column spans on each Wedderburn component by Lemma 7.8. By Lemma 7.9, we have $\Theta(e) \subseteq \Theta(\tilde{e})$ only if $\tilde{a}_\psi \in \{0, a_\psi, a_\psi + c_\psi, d_\psi\}$ for each $V \in \mathrm{Irr}(G)$ with character ψ. Note that if $c_\psi \neq 0$ then the equality $\tilde{a}_\psi = a_\psi + c_\psi$ would imply

$$0 \neq \mathbb{C}[G]\eta_H \cap \mathrm{Mat}(V) \subseteq L \cap \mathrm{Mat}(V) = \tilde{L}^\circ \cap \mathrm{Mat}(V),$$

which is a contradiction. Thus $\tilde{L}^\circ \cap \mathrm{Mat}(V) \in \{0, L^\circ \cap \mathrm{Mat}(V), \mathrm{Mat}(V)\}$ for all $V \in \mathrm{Irr}(G)$.

The converse is Lemma 7.9. □

The proof of Theorem 7.7 is now almost immediate.

Proof of Theorem 7.7. Suppose the two hypotheses of this theorem hold. By the 'if' direction of Lemma 7.10, the hypotheses imply that $\Theta(e)$ and $\Theta(\tilde{e})$ share a common Borel subalgebra. This gives the first hypothesis needed for the 'if' direction of Lemma 7.11, and the second is hypothesis (ii) in the theorem. Therefore $\Theta(e) \subseteq \Theta(\tilde{e})$. Conversely, if $\Theta(e) \subseteq \Theta(\tilde{e})$ then the 'only if' direction of Lemma 7.11 implies that $\Theta(e) \subseteq \Theta(\tilde{e})$ share a common Borel subalgebra and so the 'only if' direction of Lemma 7.10 may be applied. □

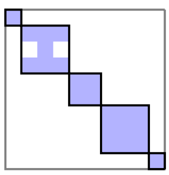
Example 7.12. Continuing Examples 3.7, 3.8, 3.19, and 7.5 we set $H = \mathrm{Sym}_3$ and $G = \mathrm{Sym}_4$. Recall that $e_{111} = \frac{1}{6}\big(1 - (12) - (13) - (23) + (123) + (132)\big)$ is the centrally primitive idempotent of $\mathbb{C}[H]$ such that $\mathbb{C}[H]e$ is the sign representation S^{111} of H. Set $e = \eta_H + e_{111}$ and $\tilde{e} = \eta_H$. Then $L = \mathbb{C}[G](\eta_H + e_{111})$ and $\tilde{L} = \mathbb{C}[G]\eta_H$. Noting that $\mathbb{1}_H = \chi^3$, Example 3.19(1) gives that

$$\mathbb{1}_H \big\uparrow_H^G = \mathbb{1}_G + \chi^{31}.$$

Thus the two representations relevant to hypothesis (i) of Theorem 7.7 are the trivial representation and S^{31}, and since

$$L^\circ = \mathbb{C}[G]e_{111} \cong S^{111}\big\uparrow_H^G \cong S^{211} \oplus S^{1111}$$

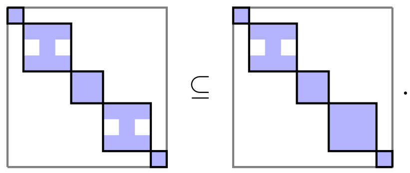
and $\tilde{L}^\circ = 0$, this hypothesis holds. Since $\tilde{L}^\circ = 0$, hypothesis (ii) of Theorem 7.7 is immediate. Therefore $\Theta(e) \subseteq \Theta(\tilde{e})$. Using the Wedderburn isomorphism introduced in Example 3.19, the algebra $\Theta(\tilde{e})$ was found in Example 7.5; the relevant diagram is repeated below.



By (7.2), we have

$$\Theta(e) = \Theta(\eta_H) \cap \mathrm{RId}(S^{111}\big\uparrow_H^G).$$

Since, as remarked in Example 3.19, our Wedderburn isomorphism is chosen so that multiplication by the sign representation of $H = \mathrm{Sym}_3$ rotates diagrams by a half-turn, the containment $\Theta(e) \subset \Theta(\tilde{e})$ is represented diagrammatically as shown below:



Corollary 7.13. *Let e, $\tilde{e} \in E^\bullet(H)$ be idempotents. Set $L = \mathbb{C}[G]e$ and $\tilde{L} = \mathbb{C}[G]\tilde{e}$. Suppose that L and $\tilde{L}$ are isomorphic as $\mathbb{C}[G]$-modules. Then, $\Theta(e) \subseteq \Theta(\tilde{e})$ if and only if $L = \tilde{L}$.*

Proof. If L and $\tilde{L}$ are isomorphic $\mathbb{C}[G]$-modules, then so are L° and $\tilde{L}^\circ$. In particular, they have the same character $\chi_{L^\circ} = \chi_{\tilde{L}^\circ}$. We deduce

$$\dim\bigl(L^\circ \cap \mathrm{Mat}(V)\bigr) = \langle \chi_{L^\circ}, \chi_V \rangle = \langle \chi_{\tilde{L}^\circ}, \chi_V \rangle = \dim\bigl(\tilde{L}^\circ \cap \mathrm{Mat}(V)\bigr)$$

for every irreducible representation $V \in \mathrm{Irr}(G)$. Now Theorem 7.7 gives the result. □

We remark that when H is non-abelian, by the remark at the end of the first paragraph of §3.4, there are infinitely many distinct idempotents affording each representation of $\mathbb{C}[H]$, and so Corollary 7.13 is a non-trivial result.

7.3. **Subgroups with full induction restriction.** We say the union defining Θ is *completely redundant* if $\Theta = \Theta(\eta_H)$, or equivalently, by Theorem 1.2, if for all $e \in E^\bullet(H)$, we have $\Theta(e) \subseteq \Theta(\eta_H)$. We say that the union is *irredundant* if a containment $\Theta(e) \subseteq \Theta(\tilde{e})$ between weak lumping algebras implies $\mathbb{C}[G]e = \mathbb{C}[G]\tilde{e}$. (Compare Corollary 7.13, that if $\mathbb{C}[G]e \cong \mathbb{C}[G]\tilde{e}$ then $\Theta(e) = \Theta(\tilde{e})$.) These two probability-theoretic properties are at opposite ends of a spectrum. In this subsection, we show they are determined by group-theoretic properties of H and G.

We begin with complete redundancy, for which we need a representation-theoretic characterisation of normality.

Lemma 7.14. *The subgroup H of G is normal if and only if $\langle \chi, \mathbb{1}\big\uparrow_H^G\big\downarrow_H \rangle = 0$ for all irreducible characters $\chi \in \mathrm{Irr}(H)$ such that $\chi \neq \mathbb{1}_H$.*

Proof. The orbits of H acting on the left on the set of left cosets G/H correspond to double cosets HxH. The stabiliser in H of xH is $H \cap xHx^{-1}$. Thinking of $\mathbb{1}\uparrow_H^G$ as the character of the permutation module (see Example 3.3) of G acting on the left cosets G/H, the character-theoretic statement of Lemma 4.2 (Mackey's Rule for the trivial character) is

$$\mathbb{1}\big\uparrow_H^G\big\downarrow_H = \sum_x \mathbb{1}_{H\cap xHx^{-1}}\big\uparrow^H$$

where the sum is over a set of representatives for the double cosets $H\backslash G/H$. If H is normal in G then the right hand side is $[G:H]\mathbb{1}_H$, as required. Otherwise there exists x such that $H \cap xHx^{-1} \neq H$, and since $\langle \mathbb{1}\big\uparrow_{H\cap xHx^{-1}}^H, \mathbb{1}_H \rangle = 1$, there exists a non-trivial irreducible character χ of H in the right-hand side. □

Proposition 7.15. *The subgroup H of G is normal if and only if the union defining Θ is completely redundant.*

Proof. By (7.2) we have

$$\Theta(\eta_H) = \mathrm{RId}_{\mathbb{C}[G]}(\mathbb{C}[G]\eta_H) \cap \mathrm{RId}_{\mathbb{C}[G]}(0) = \mathrm{RId}_{\mathbb{C}[G]}(\mathbb{C}[G]\eta_H).$$

Suppose that H is normal in G. Then η_H is central in $\mathbb{C}[G]$ (i.e. $\eta_H x = x\eta_H$ for all $x \in \mathbb{C}[G]$) and so $\mathrm{RId}_{\mathbb{C}[G]}(\mathbb{C}[G]\eta_H) = \mathbb{C}[G]$. Clearly $\mathbb{C}[G]$ contains every weak lumping algebra $\Theta(e)$.

Conversely, suppose that H is not normal in G. There exists $\chi \in \mathrm{Irr}(H)$ with $\chi \neq \mathbb{1}_H$ such that $\langle \chi, \mathbb{1}_H\!\uparrow_H^G\!\downarrow_H \rangle \neq 0$. Let U be a left ideal of $\mathbb{C}[H]$ whose character (as a $\mathbb{C}[H]$-module) is $\chi + \mathbb{1}_H$, and let $L = \mathbb{C}[G]U$. By Theorem 7.7, we deduce that $\Theta(e) \not\subseteq \Theta(\eta_H)$. □

We now consider the other end of the spectrum where the union defining Θ is irredundant, proving Proposition 1.8. The following definition is required.

Definition 7.16. We say that the subgroup H of G has *full induction restriction in G* if the restriction to H of the permutation character of G acting on the cosets of H contains every irreducible character of H. That is, if $\langle \chi, \mathbb{1}_H\!\uparrow_H^G\!\downarrow_H \rangle \neq 0$ for all $\chi \in \mathrm{Irr}(H)$.

When the group G is clear from context we abbreviate this to 'H has full induction restriction'. We use the following lemma, which may be regarded as a form of Frobenius reciprocity (see Proposition 3.20) for left ideals of group algebras.

Lemma 7.17. *Let U and $\tilde{U}$ be left ideals of $\mathbb{C}[H]$. Let V be an irreducible representation of G and let $\mathrm{Mat}(V)$ denote its Wedderburn block. Then,*

$$\mathbb{C}[G]U \cap \mathrm{Mat}(V) = \mathbb{C}[G]\tilde{U} \cap \mathrm{Mat}(V)$$

if and only if

$$U \cap \pi_H\big(\mathrm{Mat}(V)\big) = \tilde{U} \cap \pi_H\big(\mathrm{Mat}(V)\big).$$

Proof. The intersection $U \cap \tilde{U}$ is a left ideal of $\mathbb{C}[H]$. Since $\mathbb{C}[H]$ is completely reducible, it has a complement in U, call it X, and also a complement in $\tilde{U}$, call it $\tilde{X}$. Thus

$$U = (U \cap \tilde{U}) \oplus X \quad \text{and} \quad \tilde{U} = (U \cap \tilde{U}) \oplus \tilde{X},$$

where all the summands are left ideals of $\mathbb{C}[H]$. It follows that $U + \tilde{U} = (U \cap \tilde{U}) \oplus X \oplus \tilde{X}$. We have

$$\begin{aligned} \mathbb{C}[G]U \cap \mathrm{Mat}(V) &= \big(\mathbb{C}[G](U \cap \tilde{U}) \cap \mathrm{Mat}(V)\big) \oplus \big(\mathbb{C}[G]X \cap \mathrm{Mat}(V)\big) \\ = \mathbb{C}[G]\tilde{U} \cap \mathrm{Mat}(V) &= \big(\mathbb{C}[G](U \cap \tilde{U}) \cap \mathrm{Mat}(V)\big) \oplus \big(\mathbb{C}[G]\tilde{X} \cap \mathrm{Mat}(V)\big) \end{aligned}$$

and

$$\begin{aligned} \mathbb{C}[G](U + \tilde{U}) \cap \mathrm{Mat}(V) = \big(&\mathbb{C}[G](U \cap \tilde{U}) \cap \mathrm{Mat}(V)\big) \\ &\oplus \big(\mathbb{C}[G]X \cap \mathrm{Mat}(V)\big) \oplus \big(\mathbb{C}[G]\tilde{X} \cap \mathrm{Mat}(V)\big). \end{aligned}$$

But for any vector subspaces A, B, C of a common ambient vector space, if $A \oplus B = A \oplus C$ and $A \oplus B \oplus C$ is a direct sum then $B = C = \{0\}$. We deduce that

$$\mathbb{C}[G]X \cap \mathrm{Mat}(V) = 0 = \mathbb{C}[G]\tilde{X} \cap \mathrm{Mat}(V).$$

Therefore, $X \cap \pi_H\big(\mathrm{Mat}(V)\big) = X' \cap \pi_H\big(\mathrm{Mat}(V)\big) = 0$ by Frobenius reciprocity. Since $\pi_H\big(\mathrm{Mat}(V)\big)$ is a direct sum of full Wedderburn components of $\mathbb{C}[H]$, and since projection to full Wedderburn components respects direct sums of left-modules, we conclude that

$$U \cap \pi_H\big(\mathrm{Mat}(V)\big) = \tilde{U} \cap \pi_H\big(\mathrm{Mat}(V)\big) = (U \cap \tilde{U}) \cap \pi_H\big(\mathrm{Mat}(V)\big).$$

The converse is shown similarly. Indeed, from $U \cap \pi_H(\mathrm{Mat}(V)) = \tilde{U} \cap \pi_H(\mathrm{Mat}(V))$ we deduce $X \cap \pi_H(\mathrm{Mat}(V)) = 0 = \tilde{X} \cap \pi_H(\mathrm{Mat}(V))$ and the conclusion now follows from Frobenius reciprocity. □

We are now ready to prove Proposition 1.8, whose statement we recall below.

Proposition 1.8. *The subgroup H of G has full induction restriction if and only if the union defining Θ is irredundant, in the sense that no set $\Theta(e)$ is contained in another.*

Proof of Proposition 1.8. Suppose that H has full induction restriction in G. Suppose that there is a containment $\Theta(e) \subseteq \Theta(\tilde{e})$ between weak lumping algebras. Let $U = \mathbb{C}[H]e$ and $\tilde{U} = \mathbb{C}[H]\tilde{e}$. Let V be an irreducible $\mathbb{C}[H]$-submodule of $U^\circ + \tilde{U}^\circ$. Recall that χ_V denotes the character of V. Since H has full induction restriction, Frobenius reciprocity implies that $\langle \chi_V{\uparrow}_H^G, \mathbb{1}_H{\uparrow}_H^G \rangle \neq 0$. Thus Theorem 7.7 gives

$$\mathbb{C}[G]U^\circ \cap \mathrm{Mat}(V) = \mathbb{C}[G]\tilde{U}^\circ \cap \mathrm{Mat}(V).$$

We can now apply Lemma 7.17 to get

$$U^\circ \cap \pi_H(\mathrm{Mat}(V)) = \tilde{U}^\circ \cap \pi_H\big(\mathrm{Mat}(V)\big).$$

In particular, since $V \subseteq \pi_H(\mathrm{Mat}(V))$, we have $U^\circ \cap V = \tilde{U}^\circ \cap V$. This holds for all irreducible $\mathbb{C}[H]$-submodules V of $U^\circ + \tilde{U}^\circ$, and therefore $U^\circ = \tilde{U}^\circ$. Hence, $\Theta(e) = \Theta(\tilde{e})$.

For the converse, we show the contrapositive statement. Suppose that H does not have full induction restriction in G. Then there exists $\chi \in \mathrm{Irr}(H)$ such that $\langle \chi, \mathbb{1}_H{\uparrow}_H^G{\downarrow}_H \rangle = 0$. Let U be a left ideal of $\mathbb{C}[H]$ whose character (as a $\mathbb{C}[H]$-module) is $\chi + \mathbb{1}_H$, let $L = \mathbb{C}[G]U$. Note that the character of L° is $\chi{\uparrow}_H^G$ and that $\langle \chi{\uparrow}_H^G, \mathbb{1}_H{\uparrow}_H^G \rangle = 0$ by Frobenius reciprocity. Then, Theorem 7.7 gives $\Theta(e) \subseteq \Theta(\eta_H)$ and the union defining Θ is not irredundant. □

Some of the natural applications of our work are to subgroups with full induction restriction.

Example 7.18.

(1) Let $H = \mathrm{Sym}_{\{1,\ldots,k\}}$ and let $G = \mathrm{Sym}_n$. As in Examples 3.5 and 3.19, we denote by χ^λ the irreducible character of the symmetric group canonically S^λ. Then, $\langle \chi^\lambda\!\uparrow_H^G, \mathbb{1}_H\!\uparrow_H^G \rangle$ is the number of partitions of n that contain both λ and (k) as subpartitions. This constraint is strongest for $\lambda = (1^k)$; we need partitions containing $(k, 1^{k-1})$, and such partitions exist if and only if $n \geq 2k-1$. Thus H has full induction restriction if and only if $2k \leq n+1$.

(2) Let D_{2n} be the dihedral group of order $2n$, generated by σ and τ as in Example 2.1. Then $\langle \tau \rangle$ has full induction restriction and $\langle \sigma \rangle$ does not.

Both examples of full induction restriction from Example 7.18 are explained by the following lemma. For instance, in (1), if $2k \leq n-1$ then the permutation $(1, k+1)(2, k+2)\ldots(k-1, 2k-1) \in \mathrm{Sym}_n$ satisfies $g^{-1}Hg \cap H = \{1\}$, and so by (4.2), $|HgH| = |H|^2$.

Lemma 7.19. *If there is a double coset $HxH \in H\backslash G/H$ of the maximum possible size $|H|^2$, then the subgroup H of G has full induction restriction.*

Proof. By (4.2) we have $|HxH| \cdot |xHx^{-1} \cap H| = |H|^2$ for every double coset $HxH \in H\backslash G/H$. If $|HxH| = |H|^2$, we deduce $|xHx^{-1} \cap H| = 1$. Now, Mackey's rule (Lemma 4.2) gives

$$\mathbb{1}_H\!\uparrow_H^G\!\downarrow_H = \mathbb{1}_H\!\uparrow_{xHx^{-1}\cap H}^H + \cdots,$$

which by the previous computation shows that the regular representation of H appears as a summand in $\mathbb{1}_H\!\uparrow_H^G\!\downarrow_H$. □

We remark that there is a double coset HgH of the maximum possible size $|H|^2$ if and only if H has a regular orbit on the set G/H of left cosets, and so if and only if G has a base (see [11, §4.13]) of size 2 when regarded as a permutation group on G/H. The example $G = \mathrm{Sym}_6$ and $H = \langle (12)(34), (12)(3456) \rangle$ shows that full induction restriction may hold even when there is no double coset of maximum size. One can check this computationally.

8. Real idempotents and a refinement of Theorem 1.2

8.1. **Weak lumping algebras of real idempotents suffice.** Since a random walk on a group is driven by a weight whose coefficients are non-negative real numbers, to understand the set of weakly lumping weights we must consider the real parts of the algebras $\Theta(e)$ for $e \in E^\bullet(H)$. Notice that distinct weak lumping algebras may have equal real parts. For instance, take a subgroup H of order 3, with notation as in the proof of Corollary 5.17. Since complex conjugate is a multiplicative map, an elementary computation gives

$$\Theta(\xi_H + \eta_H) \cap \mathbb{R}[G] = \Theta(\overline{\xi}_H + \eta_H) \cap \mathbb{R}[G].$$

Since we are really interested in $\Delta \cap \Theta$, where Δ is the simplex of probability vectors in $\mathbb{C}[G]$, the displayed equation above shows a way in which our expression $\Delta \cap \bigcup_{e \in E^\bullet(H)} \Theta(e)$ for the set of weakly lumping weights may be redundant. (This is different to the redundancy due to containment studied in the previous section.) In this section we remedy this form of redundancy. We say that a left ideal of a complex group algebra is *self-conjugate* if it is equal to its own complex conjugate.

Lemma 8.1. *Every self-conjugate left ideal of $\mathbb{C}[H]$ is generated as a left ideal by a real idempotent of H.*

Proof. Let L be a self-conjugate left ideal of $\mathbb{C}[H]$. Consider the orthogonal projection $\pi : \mathbb{C}[H] \to L$ with respect to the Hermitian inner product on $\mathbb{C}[H]$. Because L is invariant under complex conjugation, π is equivariant with respect to complex conjugation. In particular, $\pi(h) \in \mathbb{R}[H]$ for every $h \in H$. Hence the idempotent $f = \frac{1}{|H|}\sum_{h\in H} h^{-1}\pi(h)$ is real, and $L = \mathbb{C}[H]f$, as explained in §3.4. □

Lemma 8.2. *Let $e \in E^\bullet(H)$ and let $\overline{e}$ be the complex conjugate of e. Let w be a weight on G. Then w lumps weakly with stable ideal $\mathbb{C}[G]e$ if and only if w lumps stably for the ideal $\mathbb{C}[G]\overline{e}$, and in this case w also lumps stably for the ideals $\mathbb{C}[G]e \cap \mathbb{C}[G]\overline{e}$ and $\mathbb{C}[G]e + \mathbb{C}[G]\overline{e}$. Moreover, there exist real idempotents $e^\wedge, e^\vee \in E^\bullet(H) \cap \mathbb{R}[H]$ such that $\mathbb{C}[G]e \cap \mathbb{C}[G]\overline{e} = \mathbb{C}[G]e^\wedge$ and $\mathbb{C}[G]e + \mathbb{C}[G]\overline{e} = \mathbb{C}[G]e^\vee$.*

Proof. Since complex conjugation is multiplicative, it is clear from (7.2) that $\Theta(\overline{e}) = \overline{\Theta(e)}$. We have seen that w lumps stably for $\mathbb{C}[G]e$ if and only if $w \in \Theta(e)$, and since w takes real values this holds if and only if $w \in \Theta(\overline{e})$, so if and only if w lumps stably for $\mathbb{C}[G]\overline{e}$.

We have seen in §5.1 and §5.3 that the set of weakly lumping Gurvits–Ledoux ideals for a given weight w that contain η_G forms a lattice under intersection and addition. So if w lumps stably for $\mathbb{C}[G]e$ then w also lumps stably for $\mathbb{C}[G]e\cap\mathbb{C}[G]\overline{e}$ and for $\mathbb{C}[G]e+\mathbb{C}[G]\overline{e}$. It remains to show that these two ideals (which are invariant under complex conjugation) are generated by idempotents in $E^\bullet(H) \cap \mathbb{R}[H]$.

Apply Lemma 8.1 to express $\mathbb{C}[H]e \cap \mathbb{C}[H]\overline{e} = \mathbb{C}[H]e^\wedge$ and $\mathbb{C}[H]e + \mathbb{C}[H]\overline{e} = \mathbb{C}[H]e^\vee$ for real idempotents $e^\wedge$ and $e^\vee$ in $\mathbb{C}[H]$. We have $\eta_H \in \mathbb{C}[H]e^\vee$ and $\eta_H \in \mathbb{C}[H]e^\wedge$ since $\eta_H \in \mathbb{C}[H]e$ and $\eta_H \in \mathbb{C}[H]\overline{e}$. Hence $e^\wedge, e^\vee \in E^\bullet(H) \cap \mathbb{R}[H]$. Now

$$\mathbb{C}[G]e \cap \mathbb{C}[G]\overline{e} = \mathbb{C}[G](\mathbb{C}[H]e \cap \mathbb{C}[H]\overline{e}) = \mathbb{C}[G]\mathbb{C}[H]e^\wedge = \mathbb{C}[G]e^\wedge.$$

To see the first equality, suppose that $x \in \mathbb{C}[G]e \cap \mathbb{C}[G]\overline{e}$, and pick a set $g_1, \ldots, g_m$ of coset representatives for G/H; then x may be written uniquely as $\sum g_i y_i e$ with each $y_i \in \mathbb{C}[H]$, and also uniquely as $\sum g_i z_i \overline{e}$ with each $z_i \in \mathbb{C}[H]$, and for each i we obtain $z_i\overline{e} = y_i e \in \mathbb{C}[H]e \cap \mathbb{C}[H]\overline{e}$. A similar argument shows that $\mathbb{C}[G]e + \mathbb{C}[G]\overline{e} = \mathbb{C}[G]e^\vee$. □

Corollary 8.3. *We have*

$$\Theta \cap \mathbb{R}[G] = \bigcup_{e\in E^\bullet(H)\cap\mathbb{R}[H]} \Theta(e) \cap \mathbb{R}[G].$$

Proof. Combine Theorem 1.2 with Lemma 8.2. □

It follows that the set of weakly lumping irreducible weights may be expressed as

$$\Gamma \cap \Delta \cap \Theta = \Gamma \cap \Delta \cap \bigcup_{e\in E^\bullet(H)\cap\mathbb{R}[H]} \Theta(e) \tag{8.1}$$

where, as in the introduction, $\Gamma = \mathbb{C}[G] \setminus \bigcup_{K \lneq G} \mathbb{C}[K]$ and Δ is the simplex of probability vectors in $\mathbb{C}[G]$.

8.2. **A probabilistic characterisation of stable lumping for self-conjugate left ideals.** Definition 1.3 defines in algebraic terms what it means for an irreducible weight w to lump weakly with stable ideal $L = \mathbb{C}[G]e$ for $e \in E^\bullet(H)$. As remarked after that definition, in this case it follows from Theorem 1.2 that for every weight $\alpha \in L$ we have

(a) $X = \mathrm{MC}(\alpha, w)$ lumps weakly to G/H for all $\alpha \in L$, *and*

(b) for any initial distribution $\alpha \in L$ and all t, L always contains the conditional distribution of X_t given $X_0H, \dots, X_tH$.

In this section we show a converse: for a self-conjugate left ideal $L \subseteq \mathbb{C}[G]$ containing η_G, if conditions (a) and (b) hold for all $\alpha \in \Delta \cap L$ then L is induced from $\mathbb{C}[H]$ and w lumps weakly to G/H with stable ideal L.

Lemma 8.4. *Let L be a self-conjugate left ideal of $\mathbb{C}[G]$ containing η_G. Then as a $\mathbb{C}$-vector space, L has a basis whose elements are probability vectors.*

Proof. By Lemma 8.1 (applied to G in place of H) we have $L = \mathbb{C}[G]e$ for some real idempotent $e \in E(G)$, and since $\eta_G \in L$, we have $e \in E^\bullet(G)$. Therefore L is spanned by the elements ge for $g \in G$, which are weights; some subset of these forms a basis of L. □

Proposition 8.5. *Let L be a self-conjugate left ideal of $\mathbb{C}[G]$ containing at least one non-zero probability vector and let w be an irreducible weight on G. If for every probability vector $\alpha \in L$ we have*

(i) *$X = \mathrm{MC}(\alpha, w)$ lumps weakly to G/H for all $\alpha \in L$,* and

(ii) *for any initial distribution $\alpha \in L$ and all t, L always contains the conditional distribution of X_t given $X_0H, \dots, X_tH$,*

then $L = \mathbb{C}[G]e$ for some $e \in E^\bullet(H)$, and w lumps weakly to G/H with stable ideal L.

Proof. Since w is irreducible and L contains at least one non-zero weight, by the usual ergodic averaging argument we have $\eta_G \in L$. By Lemma 8.4 we may choose a basis $v_1, \dots, v_k$ of L consisting of probability vectors. Let V denote the real vector space spanned by $v_1, \dots, v_k$. Identifying $\mathbb{C}[G]$ with $\mathbb{C}^G$, we have $V = L \cap \mathbb{R}^G$. Let P_w denote the transition matrix of the left-invariant random walk on G driven by w. By conditions (a) and (b), for all probability vectors $\alpha \in L$, applying Lemma 2.17, P_w lumps weakly under the map $\lambda : G \to G/H$ with stable space V. Thus we have $v_iP \in V$ for $i = 1, \dots, k$, so $Lw \subseteq L$. Similarly, $v_i\Pi_{bH} \in V$ for all $b \in G$ and $i = 1, \dots, k$. Extending scalars (and writing the projections to cosets on the left), this means $\pi_{bH}(L) \subseteq L$ for all $b \in G$, hence $L = \bigoplus_{b \in G/H} \pi_{bH}(L)$. Since L is also a left ideal, it is an induced ideal from the self-conjugate left ideal $\pi_H(L)$ of $\mathbb{C}[H]$, which contains η_H. Hence $L = \mathbb{C}[G]e$ for some real idempotent $e \in E^\bullet(H)$. We have $V^\circ P_w \Lambda = 0$, where Λ is the matrix for the canonical map $G \to G/H$ defined at the start of §5.2. Because Λ is real, we have $L^\circ = V^\circ \otimes_{\mathbb{R}} \mathbb{C}$, and so $L^\circ P_w \Lambda = 0$, i.e. $L^\circ w \subseteq L^\circ$. □

It follows from Proposition 5.12 that for a real idempotent $e \in E^\bullet(H) \cap \mathbb{R}[H]$, and any weight w on G, w lumps stably for $\mathbb{C}[G]e$ in the sense of Definition 1.3 if and only if P_w lumps stably for $\mathbb{R}[G]e$ in the sense of Definition 2.10. In general, not every stable space $V \subseteq \mathbb{R}[G]$ for the transition matrix P_w and the lumping map $f : G \to G/H$ need be a left ideal. However, if w is irreducible and $V \subseteq \mathbb{R}[G]$ is any subspace such that P_w lumps weakly under f with stable space V, then the space $\sum_{g \in G} gV$ is a left ideal of $\mathbb{R}[G]$ and by Lemma 2.15 it is also a stable space for P_w and f. Then $L = (\sum_{g \in G} gV) \otimes_{\mathbb{R}} \mathbb{C}$ satisfies the hypotheses of Proposition 8.5. In summary, we have proved that every stable subspace of $\mathbb{R}^G$ for weak lumping of P_w under f is a subspace of a stable ideal generated by a real idempotent in $E^\bullet(H)$.

9. Hecke algebras and the proof of Theorem 1.11

To make the proof of Theorem 1.11 self-contained, we begin by briefly reviewing the essential theory of orbital matrices and Hecke algebras. We give an example in §13.2.6 in the context of the extended dice rolling example in §13.2. For further background on Hecke algebras see [11, §1.11, §2.2, §3.1] or [12, Chapter 4]. The permutation representation of G acting on the left cosets G/H was characterised as an induced representation in Example 3.14.

9.1. **Orbital matrices.** The left action of G on the set of left cosets G/H induces an action of G on $G/H \times G/H$ by $x(gH, g'H) = (xgH, xg'H)$. Throughout this section, we set $m = |G/H|$.

Definition 9.1. Let $x \in G$. The *orbital matrix* corresponding to the double coset HxH is the $m \times m$ matrix $M(HxH)$ with zero/one entries defined by $M(HxH)_{(gH,g'H)} = 1$ if and only if $(gH, g'H)$ is in the orbit of G on $G/H \times G/H$ containing (H, xH).

Thus there is one orbital matrix for each double coset HxH and the linear span of the orbital matrices has dimension equal to the number of double cosets, namely $|H\backslash G/H|$. By the G-invariance property in the definition, each orbital matrix defines a $\mathbb{C}[G]$-endomorphism of the m-dimensional permutation representation $\mathbb{C}\uparrow_H^G$. Observe that the row of the orbital matrix $M(HxH)$ labelled by H has its non-zero entries precisely in the columns hxH for $h \in H$, corresponding to the orbit of H on G/H containing xH.

9.2. **Hecke algebras.** The Hecke algebra of functions on $\mathbb{C}[G]$ invariant under left- and right-multiplication by H is isomorphic to the subalgebra $\eta_H \mathbb{C}[G] \eta_H$ by the map sending the function $f : G \to \mathbb{C}$ to $\sum_{g \in G} f(g)g$. (Note this is the same way that we identify weights and distributions with elements of $\mathbb{C}[G]$.) It is clear that $\eta_H \mathbb{C}[G] \eta_H$ has as a basis all $\eta_H x \eta_H$ for x in a set of representatives for the double cosets $H\backslash G/H$. In particular $\dim \eta_H \mathbb{C}[G] \eta_H = |H\backslash G/H|$. We may therefore refer to $\eta_H \mathbb{C}[G] \eta_H$ as the *Hecke algebra of double cosets*.

In the following proposition, recall that the *opposite algebra* of an algebra A is the algebra with the same underlying vector space, but with multiplication defined by $a \cdot b = ba$ for a, $b \in A$.

Proposition 9.2. *For $x \in G$, let $m_x = |H/H \cap xHx^{-1}|$. The subspace of the algebra $\mathrm{Mat}_m(\mathbb{C})$ spanned by the orbital matrices is closed under multiplication and is isomorphic to the opposite algebra of $\eta_H \mathbb{C}[G]\eta_H$ by the map $\eta_H x \eta_H \mapsto M(HxH)/m_x$.*

Proof. See Proposition 4.2.1 and its proof in [12]. □

Since the proof in [12] comes only after a long development of theory not required in this paper, we outline a shorter proof: the action of $\eta_H \mathbb{C}[G]\eta_H$ by right multiplication $\mathbb{C}[G]\eta_H$ defines an injective algebra homomorphism $\eta_H \mathbb{C}[G]\eta_H \to \mathrm{End}_{\mathbb{C}[G]}(\mathbb{C}[G]\eta_H)$. By Lemma 4.2, $\dim \mathrm{End}_{\mathbb{C}[G]}(\mathbb{C}[G]\eta_H) = |H\backslash G/H|$. Hence this map is an isomorphism. In the canonical basis of left cosets of $\mathbb{C}[G]\eta_H$, the matrix of the endomorphism F_x determined by $F_x(\eta_H) = \eta_H x \eta_H$ is stochastic, having m_x entries of $1/m_x$ in each row, where $m_x = |H/H \cap xHx^{-1}|$ is the size of the orbit of H on G/H containing xH. By the final paragraph of §9.1, $\mathrm{End}_{\mathbb{C}[G]}(\mathbb{C}[G]\eta_H)$ has as a basis the orbital matrices, acting by *left* multiplication; the matrix $M(HxH)$ having m_x ones in each row corresponds to the endomorphism $m_x F_x$. Hence $\eta_H x \eta_H \mapsto M(HxH)/m_x$ is an explicit isomorphism between $\eta_H \mathbb{C}[G]\eta_H$ and the opposite algebra of the algebra of orbital matrices.

9.3. **Proof of Theorem 1.11.** We need one final preliminary: suppose that Q is an $m \times m$ matrix satisfying the condition that $Q_{(gH,g'H)} = Q_{(kgH,kg'H)}$ for all $g, g', k \in G$. By the final sentence of §9.2,

$$Q = \sum_x Q_{(H,xH)} M(HxH) \tag{9.1}$$

where the sum is over a set of representatives for the double cosets HxH.

Proof of Theorem 1.11. Suppose that (i) holds so, by hypothesis $\mathrm{MC}(\eta_G, w)$ lumps weakly to G/H and the transition matrix of the lumped chain is Q. We may assume that w is normalized, i.e. $w(G) = 1$. By hypothesis the initial distribution of X_0 is uniform. Therefore, conditioned on the event $X_0 \in gH$, the distribution of X_0 is uniform on gH. Hence for any $g' \in G$ we have

$$\begin{aligned}\mathbb{P}[X_1 \in g'H \mid X_0 \in gH] &= \frac{1}{|H|} \sum_{h \in H} \mathbb{P}[X_1 \in g'H \mid X_0 = gh] \\ &= \frac{1}{|H|} \sum_{h \in H} \sum_{h' \in H} w(h^{-1}g^{-1}g'h').\end{aligned}$$

The right hand side is the sum of the coefficients of $|H|\eta_H w \eta_H$ on the double coset $Hg^{-1}g'H$. Therefore the probability just calculated is the same replacing w with $\eta_H w \eta_H$, and we have (ii), that Q is the transition matrix of the induced random walk driven by a weight in the Hecke algebra $\eta_H \mathbb{C}[G]\eta_H$.

Suppose that (ii) holds. Then by Proposition 9.2, Q is a linear combination of the orbital matrices $M(HxH)$. It is immediate from Definition 9.1 that these matrices satisfy $M(HxH)_{(gH,g'H)} = 1$ if and only if $M(HxH)_{(kgH,kg'H)} = 1$ for all $g, g', k \in G$. Therefore (iii) holds. Conversely, if (iii) holds then by (9.1) and Proposition 9.2, a suitable weight satisfying (ii) is $\sum_x Q_{(H,xH)} \eta_H w \eta_H$.

To complete the proof it suffices to show that (ii) implies (iv) and (iv) implies (i). Suppose that (ii) holds, so that $w \in \eta_H \mathbb{C}[G]\eta_H$. Then $w(hgh') = w(g)$ for all $h, h' \in H$ and so w is constant on left cosets gH in the same double coset HxH, and dually, $w(Hg)$ is constant on right cosets Hg in the same double coset HxH. Hence w satisfies the conditions of Corollary 1.10 to lump strongly and exactly to G/H, as required for (iv). Finally (iv) implies (i) because strong lumping implies weak lumping. □

10. Duality and time-reversal

In this section we state and prove a new theorem that relates time reversal and duality for weak lumping of general finite Markov chains. We then derive Theorem 1.9 as an application, and finally deduce Corollary 1.10.

10.1. **Duality and time-reversal for weak lumpings of general finite Markov chains.** Let $X = \mathrm{MC}(\alpha, P)$ be a DTHMC, and suppose that it is stationary, i.e. X_t is distributed according to α for all times t. We may extend it to have time indexed by $\mathbb{Z}$. The time reversal of the extended chain is another stationary DTHMC which we shall denote $X^\star$. It also has time indexed by $\mathbb{Z}$ and stationary distribution α. Let $A \subseteq G$ be the support of α. Then, in the notation from the start of §2, the time reversed chain defined on A is $\mathrm{MC}(\alpha, P^\star)$, where

$$P^\star(x, y) = \frac{\alpha(y)}{\alpha(x)} P(y, x).$$

In other words, if D is the diagonal matrix indexed by A whose i^{th} diagonal entry is $\alpha(i)$, then $DP = (DP^\star)^T$. In particular, if α is the uniform distribution on A then $P^\star = P^T$.

Now suppose $f : A \to B$ is a surjective function. Then $X^\star$ lumps weakly under f if and only if X does, since if the process $f(X)$ is a time-homogeneous Markov chain (necessarily stationary), then its time reversal is also a time-homogeneous Markov chain. We shall see in Corollary 10.2 that exact lumping and strong lumping are exchanged by time reversal, under the mild condition that the stationary distribution α has full support. This is a special case of the following duality statement. See Definition 2.10 for the definition of stable space.

Theorem 10.1 (Duality). *Let α be any stationary probability distribution for P such that $\alpha(x) > 0$ for every $x \in A$. Let the time reversal of $\mathrm{MC}(\alpha, P)$ be $\mathrm{MC}(\alpha, P^\star)$. Suppose that P lumps weakly under f with stable space V, and $\alpha \in V$. If V is a real vector space, then define*

$$W = \left\{ w \in \mathbb{R}^A : \text{for all } v \in V^\circ \text{ we have} \sum_{x \in A} \frac{v(x)w(x)}{\alpha(x)} = 0 \right\},$$

and if V is a complex vector space, then define

$$W = \left\{ w \in \mathbb{C}^A : \text{for all } v \in V^\circ \text{ we have} \sum_{x \in A} \frac{\overline{v(x)}w(x)}{\alpha(x)} = 0 \right\}.$$

Then $\alpha \in W$ and the time reversal $P^\star$ lumps weakly under f with stable space W.

Proof. First let us check that $\alpha \in W$. For every $v \in V^\circ$ we have

$$\sum_{x\in A} v(x) = \sum_{b\in B} \sum_{x\in f^{-1}(b)} v(x) = \sum_{b\in B} 0 = 0,$$

as required. We now check that $P^\star$ lumps weakly under f with stable space W, by checking the four conditions in Definition 2.10.

(a) W contains the probability vector α.

(b) To show that $WP^\star \subseteq W$, let $w \in W$. We shall check that $wP^\star \in W$. For any $v \in V^\circ$ we have $vP \in V^\circ$ because $V^\circ P \subseteq V^\circ$, so

$$\begin{aligned}\sum_{x\in A} \frac{\overline{v(x)}(wP^\star)(x)}{\alpha(x)} &= \sum_{x,y\in A} \frac{\overline{v(x)}P^\star(y,x)w(y)}{\alpha(x)} \\ &= \sum_{x,y\in A} \frac{\overline{v(x)}P(x,y)w(y)}{\alpha(y)} = \sum_{y\in A} \frac{\overline{(vP)(y)}w(y)}{\alpha(y)} = 0.\end{aligned}$$

(c) To check that $W\Pi_b \subseteq W$ for all $b \in B$, note that $w \in W$ if and only if $\sum_{x\in A} \frac{\overline{v(x)}w(x)}{\alpha(x)} = 0$ for all v in a basis of V°. We may choose a basis of V° each of whose elements is supported on a single fiber of the map f. Hence this orthogonality condition holds for w if and only if it holds for $w\Pi_b$ for each $b \in B$.

(d) To show that $W^\circ P^\star \subseteq W^\circ$, where $W^\circ = W \cap \ker F$, suppose that $w \in W^\circ$. We have already checked in (b) that $wP^\star \in W$, so we must show that $wP^\star \in \ker F$. First, let us show that for every $v \in V$ we have

$$\sum_{x\in A} \frac{\overline{v(x)}w(x)}{\alpha(x)} = 0. \tag{10.1}$$

Let v' be the projection of v into V° defined by

$$v' = v - \sum_{b\in B} \frac{v(f^{-1}(b))}{\alpha(f^{-1}(b)}(\alpha\Pi_b),$$

where $v(f^{-1}(b))$ denotes $\sum_{y\in f^{-1}(b)} v(y)$ and likewise for $\alpha(f^{-1}(b))$. Then since $w \in W$ we have

$$\sum_{x\in A} \frac{\overline{v'(x)}w(x)}{\alpha(y)} = 0,$$

and so it suffices to check that for every $b \in B$ we have

$$\frac{\overline{v(f^{-1}(b))}}{\alpha(f^{-1}(b))} \sum_{x\in A} \frac{\overline{(\alpha\Pi_b)(x)}w(x)}{\alpha(x)} = \frac{\overline{v(f^{-1}(b)}}{\alpha(f^{-1}(b))} \sum_{x\in f^{-1}(b)} w(x) = 0.$$

This holds because $w \in \ker F$. We have proved (10.1).

We wish to show that $wP^\star F = 0$, in other words that for each $b \in B$ we have

$$\sum_{x\in f^{-1}(b)} (wP^\star)(x) = 0.$$

Let $b \in B$. We have

$$\begin{aligned}\sum_{x \in f^{-1}(b)} (wP^\star)(x) &= \sum_{y \in A} \sum_{x \in f^{-1}(b)} w(y) P^\star(y,x) \\ &= \sum_{y \in A} \sum_{x \in f^{-1}(b)} w(y) P(x,y) \frac{\alpha(x)}{\alpha(y)} \\ &= \sum_{y \in A} \frac{((\alpha \Pi_b) P)(y) w(y)}{\alpha(y)} \\ &= \sum_{y \in A} \frac{(\alpha \Pi_b P)(y) w(y)}{\alpha(y)}.\end{aligned}$$

This is 0 because $\overline{\alpha} = \alpha \in V$ and hence $\alpha\Pi_b \in V$ and $\alpha\Pi_b P \in V$, and so we may take $v = \overline{\alpha\Pi_b P} = \alpha\Pi_b P$ in (10.1). This concludes the check of condition (d), and the proof of Theorem 10.1.

□

The correspondence between V and W is an involution. To see this, let the inner product $\langle -, - \rangle_\alpha$ on $\mathbb{R}^A$ or $\mathbb{C}^A$ be defined by

$$\langle v, w \rangle_\alpha = \sum_{x \in A} \frac{\overline{v(x)} w(x)}{\alpha(x)}.$$

Since V and W both have $\langle \alpha\Pi_B : b \in B \rangle$ as a subspace, it follows that $V = (W^\circ)^\perp$ and $V^\circ = W^\perp$, where the orthogonal complements are with respect to the inner product $\langle -, - \rangle_\alpha$. The following special case is worth noting.

Corollary 10.2. *Let α be a stationary distribution for P with full support. Then $\mathrm{MC}(\alpha, P)$ lumps strongly under f if and only if $\mathrm{MC}(\alpha, P^\star)$ lumps exactly under f.*

Proof. Taking $V = \mathbb{R}^A$ in Theorem 10.1, we obtain $W = \langle \alpha\Pi_b : b \in B \rangle$. Dually, if we take $V = \langle \alpha\Pi_b : b \in B \rangle$ then $V^\circ = 0$ and we obtain $W = \mathbb{R}^A$. Now apply Theorem 10.1. □

10.2. **Proof of Theorem 1.9.** Recall that the anti-involution $\star$ on $\mathbb{C}[G]$ is defined by $x^\star = \sum_{g \in G} \overline{x(g)} g^{-1}$ for any element $x \in \mathbb{C}[G]$. Let $\perp$ denote the orthogonal complement with respect to the Hermitian inner product on $\mathbb{C}[G]$ defined in (1.2).

Lemma 10.3. *For any idempotent $e \in E[G]$, $(\mathbb{C}[G]e)^\perp = \mathbb{C}[G](1 - e^\star)$.*

Proof. Observe that $\langle v, w \rangle = 0$ if and only if the coefficient of the identity in $wv^\star$ is 0. Hence $ev^\star = 0$ if and only if $\langle v, ge \rangle = 0$ for all $g \in G$, if and only if $v \in (\mathbb{C}[G]e)^\perp$. Hence

$$\mathbb{C}[G]e^\perp = \{v \in \mathbb{C}[G] : ev^\star = 0\} = \{v \in \mathbb{C}[G] : ve^\star = 0\} = \mathbb{C}[G](1 - e^\star),$$

where we used that $e^\star$ is also an idempotent. □

When w is a weight, the stationary random walk $\mathrm{MC}(\eta_G, w)$ may be extended to a stationary random walk with time indexed by $\mathbb{Z}$. The time reversal of this extension is $\mathrm{MC}(\eta_G, w^\star)$, similarly extended. Thus, recalling that P_w denotes the transition matrix associated to w, we have $(P_w)^\star = P_{w^\star}$. We are now ready to prove Theorem 1.9, which we restate below.

Theorem 1.9 (Time reversal)**.** *Let $e \in E^\bullet(H)$ and let $w \in \mathbb{C}[G]$ be a weight. The left-invariant random walk on G driven by w weakly lumps to G/H with stable ideal $\mathbb{C}[G]e$ if and only if the left-invariant random walk on G driven by $w^\star$ lumps stably for $\mathbb{C}[G](1 - e^\star + \eta_H)$.*

Proof. We apply the complex case of Theorem 10.1 to $\mathrm{MC}(\eta_G, w)$. The uniform distribution $\alpha = \eta_G$ is stationary and gives positive mass to every element of $A = G$. Take $V = \mathbb{C}[G]e$. Then

$$W = (V^\circ)^\perp = (\mathbb{C}[G](e - \eta_H))^\perp = \mathbb{C}[G](1 - e^\star + \eta_H)$$

by Lemma 10.3, using that $e - \eta_H$ is an idempotent and $\eta_H^\star = \eta_H$. As mentioned in Remark 2.11, Proposition 5.12 shows that the left-invariant random walk driven by w lumps weakly to G/H with stable ideal $\mathbb{C}[G]e$ if and only if the corresponding transition matrix P_w lumps weakly with stable space $\mathbb{C}[G]e$. □

Remark 10.4. By the formula for centrally primitive idempotents (3.2), if e is a centrally primitive idempotent then $e = e^\star$. It follows that if H is abelian then, since every idempotent is then a sum of centrally primitive idempotents, we have $e = e^\star$ for all $e \in E^\bullet(H)$ and so the stable ideal for the time reversed walk is $\mathbb{C}[G](1 - e - \eta_H)$.

We are now ready to prove Corollary 1.10.

Proof of Corollary 1.10. Part (i) of Corollary 1.10 is the criterion for strong lumping that was proved in [8]. By the equivalence of (i) and (iv) in Proposition 5.15, this is equivalent to weak lumping with stable ideal $\mathbb{C}[G]$; that is, the case $e = \mathrm{id}_H$ of Theorem 1.9. Thus strong lumping occurs if and only if the left-invariant random walk on G driven by $w^\star$ lumps weakly to G/H with stable ideal $\mathbb{C}[G]\eta_H$. That is precisely the condition for the time-reversal of the stationary walk to lump exactly to G/H. Since $\star$ is an anti-involution, i.e. $(xy)^\star = y^\star x^\star$ for all x, $y \in \mathbb{C}[G]$, we see that $w(gH)$ is constant for left cosets gH in the same double coset if and only if $w^\star(Hg)$ is constant for right cosets Hg in the same double coset. Part (ii) of the corollary follows, by replacing w with $w^\star$. □

We have already seen an example of $\star$-duality and time reversal in the shuffles described in §1.2. For a deck of n cards, the random-to-top shuffle and the top-to-random shuffle are related by time reversal. Both shuffles are irreducible with the uniform distribution on Sym_n as stationary distribution. The random-to-top shuffle lumps strongly to the top card and the top-to-random shuffle lumps exactly to the top card, exemplifying Corollary 10.2. For the weight w on $G = \mathrm{Sym}_4$ defined in equation (1.3), and $H = \mathrm{Sym}_{\{2,3,4\}}$, $T = \mathrm{Sym}_{\{2,3\}}$, we obtain that

$$w^\star = (1 - \lambda)\mathrm{id} + \tfrac{\lambda}{3}\big((1,4)(2,3) + (1,3,4) + (1,3,2,4)\big)$$

lumps stably for the ideal $\mathbb{C}[G](1 - \eta_T + \eta_H)$. We shall see further examples of Theorem 1.9 when we study random rotations of a six-sided die in §13.2: see in particular Example 13.7.

11. Interpolating between strong and exact lumping

Our characterisation of weak lumping from §5 is in terms of Gurvits–Ledoux ideals (see Definition 5.11), which are induced ideals in the sense of Definition 3.15. A natural way of constructing ideals of $\mathbb{C}[H]$ is by taking ideals of $\mathbb{C}[T]$ where T is a fixed subgroup of H. Let $W = \mathbb{C}[T]\eta_T$ be the trivial representation of T, and let

$$\begin{aligned} U &= \mathbb{C}[H]\eta_T \cong W\!\uparrow_T^H, \\ L &= \mathbb{C}[G]\eta_T \cong U\!\uparrow_H^G \cong W\!\uparrow_T^G. \end{aligned}$$

Proposition 11.1. *The left ideal $L = \mathbb{C}[G]\eta_T$ is a weak lumping Gurvits–Ledoux ideal for w with respect to left cosets of H if and only if*

(a) *the random walk driven by w lumps exactly to left cosets of T, and*

(b) $w(TgH) = \dfrac{|TgH|}{|HgH|} w(HgH)$ *for all $TgH \in T\backslash G/H$.*

Proof. The weak lumping algebra of $L = \mathbb{C}[G]\eta_T$ with respect to left cosets of H is

$$\begin{aligned} \Theta(\eta_T) &= \mathrm{RId}_{\mathbb{C}[G]}(L) \cap \mathrm{RId}_{\mathbb{C}[G]}(L^\circ) \\ &= \{w \in \mathbb{C}[G] : \eta_T w(1-\eta_T) = 0, (\eta_T - \eta_H)w(1-\eta_T+\eta_H) = 0\} \\ &= \{w \in \mathbb{C}[G] : \eta_T w(1-\eta_T) = 0, (\eta_T - \eta_H)w\eta_H = 0\}. \end{aligned}$$

The first equation defines the set of *exactly* lumping weights on left cosets of T given in Example 7.4, that is

$$\{w \in \mathbb{C}[G] : \eta_T w(1-\eta_T) = 0\} = \Theta(\eta_T).$$

The second equation can be rewritten as $\eta_T w \eta_H = \eta_H w \eta_H$ and broken up coefficient by coefficient into a system of $|G|$ equations. By Lemma 4.1, we have

$$\eta_T w \eta_H = \sum_{x \in G} \left(\frac{w(TxH)}{|TxH|}\right) x \quad \text{and} \quad \eta_H w \eta_H = \sum_{x \in G} \left(\frac{w(HxH)}{|HxH|}\right) x.$$

Therefore, $\eta_T w \eta_H = \eta_H w \eta_H$ if and only if

$$w(TgH) = \frac{|TgH|}{|HgH|} w(HgH)$$

for all $g \in G$. □

We remark that the conditions in Proposition 11.1 may be rewritten as:

(a′) $w(Tg)$ is constant on $Tg \subseteq TxT$ for all $TxT \in T\backslash G/T$, and

(b′) $w(TgH)$ is constant on $TgH \subseteq HxH$ for all $HxH \in H\backslash G/H$.

We used this form of the conditions in the example in §1.2.3.

Corollary 11.2. *Let $L = \mathbb{C}[G]\eta_T$ be a Gurvits–Ledoux ideal for w with respect to left cosets of G/H.*

(i) *If $T = H$, then L is weakly lumping for w if and only if the left-invariant random walk driven by w lumps exactly to G/H.*

(ii) *If $T = 1$, then L is weakly lumping for w if and only if the left-invariant random walk driven by w lumps strongly to G/H.*

Proof. When $T = H$, condition (a) in Proposition 11.1 recovers the characterisation of exact lumping to G/H from Corollary 1.10(ii) and condition (b) becomes void. When $T = 1$, condition (b′) in the equivalent restatement above becomes the characterisation of strong lumping from Corollary 1.10(i) and (a′) becomes trivial. □

The conclusion of this corollary may hold even if T is neither 1 nor H. This is demonstrated by the following proposition.

Proposition 11.3. *Let* $G = \mathrm{Sym}_n$ *acting naturally on* $\{1, \ldots, n\}$. *Define subgroups* $H = \mathrm{Stab}(1) = \mathrm{Sym}_{\{2,\ldots,n\}}$ *and* $T = \mathrm{Stab}(1) \cap \mathrm{Stab}(n) = \mathrm{Sym}_{\{2,\ldots,n-1\}}$. *Consider the two-step shuffle of a deck of* n *cards:*

> Remove the bottom card, insert it under a random card chosen uniformly from the remaining deck, then move the top card to the bottom.

This shuffle lumps weakly to G/H *with stable ideal* $\mathbb{C}[G]\eta_T$.

Proof. Let w be the normalized weight describing the shuffle. We shall use Proposition 11.1 to show that w lumps weakly to G/H. To check that w satisfies condition (a), that $\mathrm{MC}(w, \eta_G)$ lumps exactly to G/T, we may check that the time-reversal $w^\star$ lumps strongly to G/T and apply Corollary 10.2. The shuffle described by $w^\star$ is performed as follows:

> Set aside the bottom card. Pick a card uniformly at random in the remaining deck and move it to the bottom. Put the set-aside card on top of the deck.

Each element of G/T corresponds to a particular ordered pair

(value of the top card, value of the bottom card).

The value of the top card after the next $w^\star$ shuffle is always the current value of the bottom card. Even conditional on the complete current order of the deck, the next value of the bottom card is uniform among the $n-1$ values not equal to the current value of the bottom card. Hence $w^\star$ lumps strongly to G/T, as required.

To check that w satisfies condition (b), note that there are just three double cosets in $T\backslash G/H$:

- H itself, which is also a double coset in $H\backslash G/H$; this is the set of permutations stabilising position 1;
- $(1, n)H = T(1, n)H$; this is the set of $(n-1)!$ permutations that send position n to position 1,
- $T(1, 2, n)H$; this is the set of $(n-2)(n-1)!$ permutations that send some position in $\{2, \ldots, n-1\}$ to position 1.

Under the shuffle w, position 1 is certainly sent to position n, so $w(H) = 0$. With probability $1/(n-1)$, we insert the bottom card immediately under the top card in the deck; then after the top card is moved to the bottom, the effect is that position n is sent to 1. With the remaining probability $(n-2)/(n-1)$, the card in position 2 is sent to position 1. The final two probabilities are in proportion to the sizes of the two double cosets $T(1, n)H$ and $T(1, 2, n)H$ forming the double coset $H(1, 2)H$. This establishes condition (b). □

This is an instance of Proposition 11.1 that is not covered by Corollary 11.2. Indeed it is easy to see that $w^\star$ does not lump strongly to G/H provided $n \geq 3$, and that w does not lump strongly to G/H provided $n \geq 4$. By Theorem 1.9, the shuffle $w^\star$ lumps weakly to G/H with stable ideal $\mathbb{C}[G](1-\eta_T+\eta_H)$. Note that $w^\star$ does not lump exactly to G/T when $n \geq 4$, so it does not provide another direct application of Proposition 11.1.

12. Double coset decomposition of the weak lumping algebra

Theorem 1.2 and our further results suggest that it is natural to study the problem of weak lumping to G/H 'double coset by double coset'. Corollary 1.10 and Proposition 11.1 are examples of this phenomenon. In this section, we make this explicit by proving (12.2) and then Proposition 12.10, which gives a necessary and sufficient condition for a weight w to be in $\mathbb{C}[HxH] \cap \Theta(e)$, as well as a test for this condition. In §13, we further develop these results for H abelian.

12.1. **Preliminaries.** Given $K \subseteq G$, let $\mathbb{C}[K]$ denote the subspace of $\mathbb{C}[G]$ spanned linearly by the elements $x \in K$. Given $w \in \mathbb{C}[G]$ we write w_{HxH} for the component of w in $\mathbb{C}[HxH]$; thus $w = \sum_{x \in H\backslash G/H} w_{HxH}$ is the unique expression of w as an element of $\mathbb{C}[G] = \bigoplus_{x \in H\backslash G/H} \mathbb{C}[HxH]$.

Lemma 12.1. *If $e, f \in \mathbb{C}[H]$ and $w \in \mathbb{C}[G]$ then we have $ewf = 0$ if and only if $ew_{HxH}f = 0$ for each double coset $HxH \in H\backslash G/H$.*

Proof. Write $w = \sum_{x \in H\backslash G/H} w_{HxH}$. Since $\mathbb{C}[HxH]$ is a left $\mathbb{C}[H]$-module (by left multiplication) and a right $\mathbb{C}[H]$-module (by right multiplication), we have $ewf = 0$ if and only if $ew_{HxH}f = 0$ for each double coset HxH. □

12.2. **The module structure of $\mathbb{C}[HxH]$.** In order to use representation theoretic tools to study $\mathbb{C}[HxH]$, we require a version of Mackey's rule (given earlier in Lemma 4.2 in a special case) with an explicit isomorphism for each double coset. For this, we need to understand twisted actions. Induced modules give the most important examples.

Example 12.2. Observe that in $U\uparrow_H^G = \bigoplus_{x \in G/H} \langle x \rangle \otimes U$, the subspace $\langle x \rangle \otimes U$ is a xHx^{-1}-module on which $k = xhx^{-1} \in xHx^{-1}$ acts by

$$k(x \otimes v) = x(x^{-1}xg) \otimes u = x \otimes x^{-1}kxu.$$

Definition 12.3. Let H be a subgroup of G, and $y \in G$. Given a left $\mathbb{C}[H]$-module M we denote by ${}^xM$ the left $\mathbb{C}[xHx^{-1}]$-module with the same underlying vector space as M but with the action $xhx^{-1} \cdot v = hv$ for all $h \in H$ and $v \in M$.

Note that in this definition hv is defined using the original $\mathbb{C}[H]$-module structure on M. Equivalently, as expected from Example 12.2,

$$k \cdot v = x^{-1}kxv \tag{12.1}$$

for $k \in xHx^{-1}$ and $v \in M$. Thus we have $\chi_{{}^xM}(xhx^{-1}) = \chi_M(h)$, or equivalently, $\chi_{{}^xM}(k) = \chi_M(x^{-1}kx)$. This is, by definition, the conjugated character $\chi_M^{x^{-1}}$. (The inverse is necessary to be consistent with the usual notation, see for instance [21, Ch. 6].)

In the following lemma we need the natural right action of $\mathbb{C}[H]$ on $\mathbb{C}[HxH]$ defined by $hxh' \cdot k = hxh'k$, and on ${}^x\mathbb{C}[H]$ defined by $h' \cdot k = h'k$. It is routine to check from (4.1) that these right actions on $\mathbb{C}[HxH]$ are well defined.

Lemma 12.4. *There is an isomorphism of left $\mathbb{C}[H]$-modules*

$$\mathbb{C}[HxH] \cong \mathbb{C}[H] \otimes_{\mathbb{C}[H\cap xHx^{-1}]} {}^x\mathbb{C}[H]$$

defined by $hxh' \mapsto h \otimes h'$. Moreover this map commutes with the natural right actions of $\mathbb{C}[H]$ on $\mathbb{C}[HxH]$ and on ${}^x\mathbb{C}[H]$.

Proof. Clearly $\mathbb{C}[HxH]$ contains the subspace W spanned by the set $(H \cap xHx^{-1})xH$. Since $xhx^{-1}x = xh$, the subspace W has $\{xh : h \in H\}$ as a canonical basis. Observe that W is a left $\mathbb{C}[H \cap xHx^{-1}]$-module on which $k \in H \cap xHx^{-1}$ acts by permuting the canonical basis:

$$kxh' = x(x^{-1}kx)h'.$$

Thus, by (12.1), W is isomorphic to $({}^x\mathbb{C}[H])\big\downarrow_{H\cap xHx^{-1}}$ as a module for $H \cap xHx^{-1}$. Let $m = [H : H \cap xHx^{-1}]$ and let $h_1, \ldots, h_m$ be representatives for the left cosets $H/H \cap xHx^{-1}$. Each $g \in HxH$ may be expressed uniquely as $g = h_i xh$ for some $h \in H$. Hence $\mathbb{C}[HxH] = \bigoplus_{i=1}^m h_i W$. It now follows from the characterisation of induced modules of Proposition 3.13 and the definition of induction that $\mathbb{C}[HxH] \cong \mathbb{C}[H] \otimes_{\mathbb{C}[H\cap xHx^{-1}]} W$. An explicit isomorphism of left $\mathbb{C}[H]$-modules is defined by $h_i v \mapsto h_i \otimes v$ for $1 \le i \le m$ and $v \in W$. Equivalently, $hxh' \mapsto h \otimes h'$ for $h, h' \in H$. It is routine to check that this isomorphism commutes with the right action of $\mathbb{C}[H]$. □

Proposition 12.5. *Let $f \in \mathbb{C}[H]$. There is an isomorphism of left $\mathbb{C}[H]$-modules*

$$\mathbb{C}[HxH]f \cong \big({}^x(\mathbb{C}[H]f)\big\downarrow_{H\cap xHx^{-1}}\big)\big\uparrow^H$$

defined by $hxh'f \mapsto h \otimes h'f$.

Proof. Take the isomorphism in Lemma 12.4 and multiply each side on the right by f, using that the isomorphism commutes with the right action of $\mathbb{C}[H]$, to get

$$\mathbb{C}[HxH]f \cong \mathbb{C}[H] \otimes_{\mathbb{C}[H\cap xHx^{-1}]} {}^x(\mathbb{C}[H]f),$$

where the isomorphism is defined by $hxh'f \mapsto h \otimes h'f$. By definition, the right-hand side is the induced module in the proposition. □

12.3. **Double coset decomposition of right idealizers.** Let e be an idempotent and $L = \mathbb{C}[G]e$ be a left ideal of $\mathbb{C}[G]$. From Lemma 12.1, the conditions $ew(1-e) = 0$ defining the right idealizer of L reduce to conditions on each double coset. Denoting $\mathrm{RId}_{\mathbb{C}[G]}(L) \cap \mathbb{C}[HxH]$ by $\mathrm{RId}_{\mathbb{C}[HxH]}(L)$, we have

$$\mathrm{RId}_{\mathbb{C}[G]}(L) = \bigoplus_{x\in H\backslash G/H} \mathrm{RId}_{\mathbb{C}[HxH]}(L).$$

(To be very careful, we warn the reader that the space $\mathrm{RId}_{\mathbb{C}[HxH]}(L)$ is not a right idealizer in the usual sense of §3.7, because $L \not\subseteq \mathbb{C}[HxH]$.)

Proposition 12.6. *Fix $x \in G$ and let $e \in \mathbb{C}[H]$ be an idempotent $e \in \mathbb{C}[H]$. Then*

$$\dim\big(\mathrm{RId}_{\mathbb{C}[HxH]}(L)\big) = |HxH| - \big\langle \chi_{\mathbb{C}[H]e}\!\downarrow_{H\cap xHx^{-1}}, (\chi^{x^{-1}}_{\mathbb{C}[H](1-e)})\!\downarrow_{H\cap xHx^{-1}}\big\rangle$$

Proof. We apply Proposition 12.5 and Lemma 3.10 to the left $\mathbb{C}[H]$-module $\mathbb{C}[HxH](1-e)$ to obtain

$$\begin{aligned}\dim e\mathbb{C}[HxH](1-e) &= \big\langle \chi_{\mathbb{C}[H]e}, (\chi^{x^{-1}}_{\mathbb{C}[H](1-e)})\!\downarrow_{H\cap xHx^{-1}}\!\uparrow^{H}\big\rangle \\ &= \big\langle \chi_{\mathbb{C}[H]e}\!\downarrow_{H\cap xHx^{-1}}, (\chi^{x^{-1}}_{\mathbb{C}[H](1-e)})\!\downarrow_{H\cap xHx^{-1}}\big\rangle\end{aligned}$$

where the second line follows from Frobenius reciprocity. Now apply Lemma 3.24. □

We remark that it is useful to interpret the equation in the proposition as giving, in the quantity subtracted, the number of linear equations that define the algebra $\{w : ew(1-e) = 0\}$ on the double coset HxH; equivalently this is the codimension of the direct summand of the algebra in $\mathbb{C}[HxH]$. We now explore consequences of this proposition. A notable case is when no equations are required.

Corollary 12.7. *There is no constraint on an element $w \in \mathbb{C}[G]$ satisfying $ew(1-e) = 0$ from the double coset HxH if and only if either*

$$\big\langle \chi_{\mathbb{C}[H]e}\!\downarrow_{H\cap xHx^{-1}}, (\chi^{x^{-1}}_{\mathbb{C}[H](1-e)})\!\downarrow_{H\cap xHx^{-1}}\big\rangle = 0$$

or the same condition holds swapping e and $1-e$.

Proof. This follows from Proposition 12.6 and the interpretation made immediately above. □

In practice it is more convenient to have a condition using just one character.

Corollary 12.8. *Let $x \in G$ and let $w \in \mathbb{C}[HxH]$. Let $e \in \mathbb{C}[H]$ be an idempotent and define $L = \mathbb{C}[G]e$, $U = \mathbb{C}[H]e$ and $U^{\mathrm{c}} = \mathbb{C}[H](1-e)$. Then*

$$\operatorname{codim}\big(\mathrm{RId}_{\mathbb{C}[HxH]}(L)\big) = \frac{|HxH|}{|H|}\phi(1) - \big\langle \phi\!\downarrow_{H\cap xHx^{-1}}, \phi^{x^{-1}}\!\downarrow_{H\cap xHx^{-1}}\big\rangle$$

where ϕ is either χ_U or $\chi_{U^{\mathrm{c}}}$.

Proof. Recall from Example 3.2 that the character of the regular representation of a group K is ρ_K, defined by $\rho_K(1) = |K|$ and $\rho_K(k) = 0$ if $k \neq 1$. Since $\mathbb{C}[H] = U \oplus U^{\mathrm{c}}$, we have $\chi_U + \chi_{U^{\mathrm{c}}} = \rho_H$. Setting $\phi = \chi_U$, Proposition 12.6 implies that

$$\begin{aligned}&\operatorname{codim}\big(\mathrm{RId}_{\mathbb{C}[HxH]}(L)\big) \\ &\quad= \big\langle \phi\!\downarrow_{H\cap xHx^{-1}}, (\rho_H - \phi)^{x^{-1}}\!\downarrow_{H\cap xHx^{-1}}\big\rangle \\ &\quad= \big\langle \phi\!\downarrow_{H\cap xHx^{-1}}, \rho_H\!\downarrow_{H\cap xHx^{-1}}\big\rangle - \big\langle \phi\!\downarrow_{H\cap xHx^{-1}}, \phi^{x^{-1}}\!\downarrow_{H\cap xHx^{-1}}\big\rangle \\ &\quad= \frac{\phi(1)|H|}{|H\cap xHx^{-1}|} - \big\langle \phi\!\downarrow_{H\cap xHx^{-1}}, \phi^{x^{-1}}\!\downarrow_{H\cap xHx^{-1}}\big\rangle.\end{aligned}$$

Since $|H|/|H\cap xHx^{-1}| = |HxH|/|H|$ by (4.2) this gives the first case when $\phi = \chi_U$, and the second case when $\phi = \chi_{U^{\mathrm{c}}}$ is proved by replacing U with U^{c} throughout. □

In particular, the condition in Corollary 12.7 holds whenever *both* $xHx^{-1} = H$, *and* χ_U is invariant under conjugation by x. On the other hand, *it never* holds when HxH is a double coset of maximum possible size. Indeed in this case, since $|HxH| = |H|^2$, we have

$$\begin{aligned} \dim\big(\mathrm{RId}_{\mathbb{C}[HxH]}(L)\big) &= \dim\,\{w \in \mathbb{C}[HxH] : ew(1-e) = 0\} \\ &= |H|^2 - \dim U^{\mathrm{c}} \cdot \dim U. \end{aligned}$$

Example 12.9 (Exact lumping)**.** By Proposition 5.16(iv), the left-invariant random walk driven by w lumps exactly if and only if $w \in \Theta(\eta_H)$. By counting the number of linear constraints imposed by Corollary 1.10 we find that the codimension of $\Theta(\eta_H) \cap \mathbb{C}[HxH]$ in $\mathbb{C}[G]$ is $|HxH|/|H| - 1$. Note this is one less than the number of left cosets forming HxH. It is instructive to reprove this result using Corollary 12.8: setting $\phi = \mathbb{1}_H$, the codimension of $\Theta(\eta_H) \cap \mathbb{C}[HxH]$ is

$$\frac{|HxH|}{|H|}\mathbb{1}_H(\mathrm{id}_H) - \langle \mathbb{1}_{H\cap xHx^{-1}}, \mathbb{1}_{H\cap xHx^{-1}} \rangle = \frac{|HxH|}{|H|} - 1$$

as just seen.

We leave it to the interested reader to formulate the analogous example for strong lumping; the codimension in each double coset is the same, as should be expected from the $\star$-duality in Theorem 1.9. Indeed, $\Theta(\mathbb{1}_H) = \Theta(\eta_H)^\star$ and the $\star$ operation respects the direct sum decomposition of these algebras over double cosets in (12.2), exchanging the summands for $\mathbb{C}[HxH]$ and $\mathbb{C}[Hx^{-1}H]$.

12.4. **Double coset decomposition of weak lumping algebras.** Let $e \in E^\bullet(H)$ be an idempotent, let $L = \mathbb{C}[G]e$. The double coset decompositions of $\mathrm{RId}_{\mathbb{C}[G]}(L)$ and $\mathrm{RId}_{\mathbb{C}[G]}(L^\circ)$ give in turn a decomposition of the weak lumping algebra of e,

$$\Theta(e) = \bigoplus_{x \in H\backslash G/H} \Theta(e) \cap \mathbb{C}[HxH] \tag{12.2}$$

and so weak lumping of irreducible weights is decided double coset by double coset.

Proposition 12.10. *Let $w \in \mathbb{C}[HxH]$. Let $e \in E^\bullet(H)$ be an idempotent, and set $L = \mathbb{C}[G]e$. A necessary and sufficient condition for $w \in \Theta(e) \cap \mathbb{C}[HxH]$ is that $w \in \mathrm{RId}_{\mathbb{C}[HxH]}(L^\circ)$ and there exists $v \in L^\circ$ such that $w(Hxh) + v(Hxh)$ is constant for $h \in H$. Moreover if such a v exists, then $\eta_H v$ has the same property.*

More informally we may say 'w is constant on right cosets modulo L°'.

Proof. Since $\Theta(e) \cap \mathbb{C}[HxH] = \mathrm{RId}_{\mathbb{C}[HxH]}(L^\circ) \cap \mathrm{RId}_{\mathbb{C}[HxH]}(L)$ we may assume throughout that $w \in \mathrm{RId}_{\mathbb{C}[HxH]}(L^\circ)$. Since $L = L^\circ \oplus \mathbb{C}[G]\eta_H$ we have, under this assumption,

$$\begin{aligned} &w \in \Theta(e) \cap \mathbb{C}[HxH] \\ &\iff \mathbb{C}[G]\eta_H w \subseteq (L^\circ \oplus \mathbb{C}[G]\eta_H) \cap \mathbb{C}[HxH] \\ &\iff \eta_H w \in (L^\circ \cap \mathbb{C}[HxH]) \oplus \langle hx\eta_H : h \in H\rangle \\ &\iff \eta_H w \in \eta_H(L^\circ \cap \mathbb{C}[HxH]) \oplus \eta_H\langle hx\eta_H : h \in H\rangle \\ &\iff \eta_H w = \eta_H v + c\eta_H x \eta_H \text{ for some } v \in L^\circ \cap \mathbb{C}[HxH] \text{ and } c \in \mathbb{C}, \end{aligned}$$

where the third double implication follows by multiplying on the left by the idempotent η_H. Now observe that by Lemma 4.1(i) and (iii), the condition in the final line holds if and only if there exist $v \in L^\circ \cap \mathbb{C}[HxH]$ and $c \in \mathbb{C}$ such that

$$\frac{w(Hxh)}{|H|} = \frac{v(Hxh)}{|H|} + \frac{c}{|HxH|}$$

for all $h \in H$. Since $L^\circ = \bigoplus_{x \in H\backslash G/H}(L^\circ \cap \mathbb{C}[HxH])$, it is equivalent to require $v \in L^\circ$; this proves the condition is necessary and sufficient. Finally, by Lemma 4.1(i), we have $(\eta_H v)(Hg) = v(Hg)$ for each g, so we may replace v with $\eta_H v$. □

Example 12.11 (Strong lumping)**.** Let $e = 1$ and $L = \mathbb{C}[G]$. Thus $L^\circ = \mathbb{C}[G](1 - \eta_H) = \mathrm{Ann}_{\mathbb{C}[G]}(\eta_H)$ is the space of all $w \in \mathbb{C}[G]$ whose sum is zero on each left coset gH. By Lemma 4.1(ii), $\eta_H\mathbb{C}[G](1-\eta_H)$ is the space of all elements of $\mathbb{C}[G]$ constant on each right coset Hg having zero sum on each left coset gH.

For instance, the left diagram overleaf shows the elements in a double coset HxH for $H = \langle h\rangle \cong C_6$ with $xHx^{-1} \cap H = \langle h^3\rangle$; thus each $hxH \cap Hxh'$ has the form $\{g, h^3g\}$ where $h^3g = gh^3$. On the right we show a general $w \in \eta_H\mathbb{C}[G](1 - \eta_H)$, where the condition "$w(g) = a$ for all $g \in xH \cap Hx$" is represented by placing an a in the cell in column xH and row xH of the table, and similarly for all other cells. Note that since w is constant on right cosets, it is in particular constant on each $hxH \cap Hxh'$.

	Hx	Hxh	Hxh^2
xH	x, h^3x	xh, h^3xh	xh^2, h^3xh^2
hxH	hx, h^4x	hxh, h^4xh	hxh^2, h^4xh^2
h^2xH	h^2x, h^5x	h^2xh, h^5xh	h^2xh^2, h^5xh^2

Hx	Hxh	Hxh^2
a	b	c
a	b	c
a	b	c

$a + b + c = 0$

Therefore given any $\eta_H w$, necessarily constant on right cosets, we may add an element of $\eta_H L^\circ$, of the form shown above, to obtain an element of $\mathbb{C}[G]$ constant on HgH. This verifies the necessary and sufficient condition in Proposition 12.10, and shows that the condition is equivalent to $w \in \Theta(1)$. This is of course expected as $L = \mathbb{C}[G]$ and so the requirement $Lw \subseteq L$ imposes no constraints.

For a further example of Proposition 12.10 see Remark 13.6.

13. Abelian subgroups

In this section we consider the case where H is abelian. In this case Theorem 1.2 can be made very explicit. The irreducible representations of an abelian group H are all 1-dimensional and may be identified with its irreducible characters, forming the group $\widehat{H}$. The irreducible representation corresponding to the character β is afforded by the idempotent element

$$e_\beta = \frac{1}{|H|}\sum_{h\in H}\beta(h^{-1})h. \tag{13.1}$$

Note that the inverse is necessary so that we have

$$he_\beta = \beta(h)e_\beta \tag{13.2}$$

for each $h \in H$. The following lemma records some basic properties we need. Note that in (ii), the perpendicular space is taken with respect to the canonical H-invariant inner product on $\mathbb{C}[H]$, defined by taking $H = G$ in (1.2).

Lemma 13.1. *Let H be an abelian group, and let β, $\gamma \in \widehat{H}$.*

(i) *If $\beta, \gamma \in \widehat{H}$ then $e_\beta e_\gamma = e_\gamma e_\beta = 0$.*
(ii) *For each $\beta \in \widehat{H}$ we have $\langle e_\beta\rangle^\perp = \langle e_\gamma : \gamma \neq \beta\rangle$.*
(iii) *We have $1 = \sum_{\beta\in\widehat{H}} e_\beta$.*
(iv) *Every primitive idempotent of H is of the form e_β for some $\beta \in \widehat{H}$.*
(v) *We have $x^{-1}e_\beta x = e_{\beta^x}$.*

Proof. Parts (i) and (ii) follow from orthogonality of characters; parts (iii) and (iv) can be seen from the Wedderburn decomposition noting that every block is one-dimensional. Part (v) is most simply proved by calculation:

$$\begin{aligned} e_\beta x &= \frac{1}{|H|}\sum_{h\in H}\beta(h^{-1})hx = \frac{1}{|H|}\sum_{h\in H}\beta(h^{-1})x(x^{-1}hx) \\ &= \frac{1}{|H|}\sum_{k\in H}\beta(xk^{-1}x^{-1})xk = \frac{1}{|H|}\sum_{k\in H}\beta^k(x)xk = xe_{\beta^x} \end{aligned}$$

where the penultimate equality holds since $\beta(xk^{-1}x^{-1}) = \beta^x(k^{-1})$. □

Given a subset $P \subseteq \widehat{H}$ of irreducible characters of H, let

$$e_P = \sum_{\beta\in I} e_\beta.$$

By Lemma 13.1(iv), every idempotent in $\mathbb{C}[H]$ is of the form e_P, and so the left $\mathbb{C}[G]$-modules K we must consider are precisely those $\mathbb{C}[G]e_P$ as I ranges over $\widehat{H}\backslash\{\mathbb{1}_H\}$. Note that there are only finitely many such modules; in fact this feature characterises the case of abelian H.

13.1. **Double coset decomposition of weak lumping algebras: the case of abelian H.** Given a subset W of $\mathbb{C}[HxH]$, let $W^\perp$ denote the perpendicular space to W *inside* $\mathbb{C}[HxH]$, with respect to the canonical G-invariant inner product on $\mathbb{C}[G]$ defined in (1.2).

Proposition 13.2. *Let H be an abelian subgroup of G. For $P \subseteq \widehat{H}$ we have*

$$\mathrm{RId}_{\mathbb{C}[HxH]}(\mathbb{C}[G]e_P)^\perp = \langle e_\beta x e_\gamma : \beta \in P, \gamma \in \widehat{H}\backslash P\rangle.$$

Proof. By Lemmas 3.24 and 12.1 we can write

$$\mathrm{RId}_{\mathbb{C}[HxH]}(\mathbb{C}[G]e_P) = \mathbb{C}[HxH]e_P + (1-e_P)\mathbb{C}[HxH].$$

The rest of this proof follows from parts (i)–(v) of Lemma 13.1. By part (i), the image of right multiplication by e_P is

$$\langle hxke_\beta : \beta \in P, h, k \in H\rangle.$$

By (13.2) we may simplify this to $\langle hxe_\beta : \beta \in P, h \in H\rangle$. Now by part (ii) the perpendicular space of the image is $\langle hxe_\gamma : \gamma \notin P, h \in H\rangle$. Similarly, the image of left multiplication by $1-e_P$ is $\langle e_\gamma xk : \gamma \in \widehat{H}\backslash P, k \in H\rangle$ and its perpendicular space is $\langle e_\beta xk : \beta \in P, k \in H\rangle$. Taking the intersection of the perpendicular spaces gives the result. □

We now obtain the analogue of Proposition 12.10 for the case of abelian H. Recall that $\Theta(e)$, as defined in (1.1), is the algebra of weakly lumping weights for the idempotent $e \in E^\bullet(H)$ and by (12.2) we have $\Theta(e) = \bigoplus_{x\in H\backslash G/H} \Theta(e) \cap \mathbb{C}[HxH]$, and so, as we noted after this equation, weak lumping of weights is decided double coset by double coset. As further motivation for the hypothesis below, note that $e_P \in E^\bullet(H)$ if and only if $\mathbb{1}_H \in P$.

Corollary 13.3. *Let H be an abelian subgroup of G. Let $P \subseteq \widehat{H}$ contain $\mathbb{1}_H$ and let $w \in \mathbb{C}[HxH]$. A necessary and sufficient condition for $w \in \Theta(e_P)$ is that*

$$w \in \big(\langle e_\beta x e_\gamma : \beta \in P, \gamma \in \widehat{H}\backslash P\rangle + \langle e_\beta x \eta_H : \beta \in P\backslash\{\mathbb{1}_H\}\rangle\big)^\perp.$$

Proof. By (7.2) we have $\Theta(e) = \mathrm{RId}_{\mathbb{C}[G]}(\mathbb{C}[G]e_P) \cap \mathrm{RId}_{\mathbb{C}[G]}(e_P - \eta_H)$. By Proposition 13.2 we have

$$\mathrm{RId}_{\mathbb{C}[HxH]}(\mathbb{C}[G]e_P)^\perp = \langle e_\beta x e_\gamma : \beta \in P, \gamma \in \widehat{H}\backslash P\rangle$$
$$\mathrm{RId}_{\mathbb{C}[HxH]}(\mathbb{C}[G](e_P-\eta_H))^\perp = \langle e_\beta x e_\gamma : \beta \in I\backslash\{\mathbb{1}_H\}, \gamma \in (\widehat{H}\backslash I) \cup \{\mathbb{1}_H\}.$$

Therefore $w \in \Theta(e) \cap \mathbb{C}[HxH]$ if and only if it is in the first perpendicular space, and also perpendicular to all $e_\beta x \eta_H$ for $\beta \in I\backslash\{\mathbb{1}_H\}$, as required. □

Applying the corollary to each double coset in turn we obtain finitely many linear equations that specify a necessary and sufficient condition for a weight w to lie in the algebra $\Theta(e)$. In the extreme case when there is a coset $|HxH| = |H|^2$ of maximum size, the elements $e_\beta x e_\gamma$ are linearly independent and there are $|P|(|H|-|P|)$ linearly independent equations from the coset HxH. In general, since each element of HxH has $|H \cap xHx^{-1}|$ different expressions in the form hxh' for $h, h' \in H$, some work is needed to get an irredundant set of equations.

We are now ready to prove our final main result, Corollary 1.12, which we restate below for convenience.

Corollary 1.12. *Let D be a set of double coset representatives for $H\backslash G/H$. The left-invariant random walk on G driven by an irreducible weight w lumps weakly on the left cosets of H if and only if there exists a subset $P \subseteq \widehat{H}$ containing $\mathbb{1}_H$ such that for all $x \in D$ we have $w \in \bigcap_{x\in X} W_x^\perp$, where*

$$W_x = \big\langle e_\beta x e_\gamma :\ \beta \in P,\ \gamma \in (\widehat{H}\backslash P) \cup \{\mathbb{1}_H\},\ (\beta,\gamma) \neq (\mathbb{1}_H, \mathbb{1}_H)\big\rangle.$$

Proof. This follows by applying Corollary 13.3 to each double coset in turn using (12.2). Note that $W_x^\perp$ is the perpendicular space in this corollary. □

13.1.1. *Cosets of the normalizer.* In one important case this difficulty does not arise. Recall from Definition 12.3 that if $x \in G$ and $\beta \in \widehat{H}$ then β^x denotes the character of $x^{-1}Hx$ defined by $\beta^x(k) = \beta(xkx^{-1})$. Given $P \subseteq \widehat{H}$ we write P^x for $\{\beta^x : \beta \in P\}$. As motivation, we remark that the condition $xH = Hx$ holds if and only $xH = Hx = HxH$ and if $xHx^{-1} = H$, and so if and only if x is in the normalizer $N_G(H)$; in this case $HxH = xH = Hx$.

Corollary 13.4. *Let H be an abelian subgroup of G. Let $P \subseteq \widehat{H}$ contain $\mathbb{1}_H$ and let $w \in \mathbb{C}[HxH]$ be a weight. Suppose that $xH = Hx$. Then a necessary and sufficient condition for $w \in \Theta(e_P)$ is that*

$$w \in \big\langle xe_\delta : \delta \in P^x \cap (\widehat{H} \setminus P)\big\rangle^\perp.$$

Proof. By Lemma 13.1(v) we have $e_\beta x = xx^{-1}e_\beta x = xe_{\beta^x}$. By this observation and Lemma 13.1(i) we have

$$e_\beta x e_\gamma = \begin{cases} e_{\beta^x} & \text{if } \beta^x = \gamma \\ 0 & \text{otherwise.} \end{cases}$$

Now apply this to the sum of the two perpendicular spaces given in Corollary 13.3, noting that $e_{\beta^x}\eta_H = 0$ by Lemma 13.1(i), since $\beta^x \neq \mathbb{1}_H$. □

In particular, if $x \in H$, or more generally, if $Hx = xH$ *and* the conjugacy action x permutes the irreducible characters in P, then there is no constraint from the double coset HxH on the weights in $\Theta(e) \cap \mathbb{C}[HxH]$.

13.2. **An extended example: the six-sided die.** Consider an ordinary six-sided die, that is rolled and then translated to its original position. For instance, imagine it is in an automatic dice roller which only allows for one stable position of the die. The orientation-preserving symmetries of a cube are realised by the symmetric group $G = \mathrm{Sym}_4$, acting by permuting the four diagonals of the cube. An observation of the top value of a die is invariant under the group of symmetries of the top face, which is the cyclic group $H = \langle(1,2,3,4)\rangle = \langle h\rangle \cong C_4$, where $h = (1,2,3,4)$. Therefore repeated observations of the top face correspond to lumping of the left-invariant random walk on G to the left cosets G/H.

13.2.1. *Face action.* It is useful to understand the action on the faces. Recall that opposite faces sum to 7. If the top face is ⚀, we set the convention that the permutation $(1,2,3,4)$ acts on the faces as the permutation (⚁, ⚂, ⚄, ⚃) sending the face ⚁ to ⚂, the face ⚂ to ⚄, and so on. We let the permutation $(1,2)$ act by (⚀, ⚁)(⚂, ⚃)(⚄, ⚅). This permutation swaps two diagonals of the cube, and corresponds to the 180° rotation about the axis through the centre of the edge between faces ⚀ and ⚁ and the centre of the edge between faces ⚄ and ⚅.

13.2.2. *Idempotents.* The subgroup H is abelian. Its four primitive idempotents are

$$\begin{aligned} e_{\mathbb{1}} = \eta_H = \eta &= \tfrac{1}{4}(1 + h + h^2 + h^3),\\ \xi &= \tfrac{1}{4}(1 + ih - h^2 - ih^3),\\ e_{\mathrm{sgn}} = \varsigma &= \tfrac{1}{4}(1 - h + h^2 - h^3),\\ \bar{\xi} &= \tfrac{1}{4}(1 - ih - h^2 + h^3). \end{aligned}$$

Hence $E^\bullet(H)$ has 8 elements, obtained by taking η plus one of the possible combinations of the other idempotents. By (1.1), w is a weak lumping weight if and only if $w \in \bigcup_{e \in E^\bullet(H)} \Theta(e)$; by Corollary 13.3 this is decided double coset by double coset by finitely many linear equations. Among the 8 idempotents, only the 4 which have either both or neither of ξ and $\bar{\xi}$ as a summand are real idempotents, as studied in §8; we compute $\Theta(e)$ for each of these.

13.2.3. *Double coset decomposition.* There are three double cosets in $H\backslash G/H$, namely H, $H(1,3)H$ and $H(1,2)H$, where $H(1,3)H = H(1,3) = (1,3)H$ is of the type in Corollary 13.4. The double coset $H(1,2)H$ of size 16 controls which weights lump weakly to G/H. Its structure is shown in Figure 3.

	$H(1,2)$	$H(1,2)h^2$	$H(1,2)h$	$H(1,2)h^3$
$(1,2)H$	$(1,2)$ (⚀,⚁)(⚂,⚃)(⚄,⚅)	$(1,4,2,3)$ (⚀,⚄,⚅,⚁)	$(1,3,4)$ (⚀,⚂,⚁)(⚃,⚄,⚅)	$(2,4,3)$ (⚀,⚃,⚁)(⚂,⚄,⚅)
$h^2(1,2)H$	$(1,3,2,4)$ (⚀,⚁,⚅,⚄)	$(3,4)$ (⚀,⚄)(⚁,⚅)(⚂,⚃)	$(1,4,2)$ (⚀,⚂,⚄)(⚁,⚅,⚃)	$(1,2,3)$ (⚀,⚃,⚄)(⚁,⚅,⚂)
$h(1,2)H$	$(2,3,4)$ (⚀,⚁,⚃)(⚂,⚅,⚄)	$(1,3,2)$ (⚀,⚄,⚃)(⚁,⚂,⚅)	$(1,2,4,3)$ (⚀,⚂,⚅,⚃)	$(1,4)$ (⚀,⚃)(⚁,⚄)(⚂,⚅)
$h^3(1,2)H$	$(1,4,3)$ (⚀,⚁,⚂)(⚃,⚅,⚄)	$(1,2,4)$ (⚀,⚄,⚂)(⚁,⚃,⚅)	$(2,3)$ (⚀,⚂)(⚁,⚄)(⚃,⚅)	$(1,3,4,2)$ (⚀,⚃,⚅,⚂)

	$H(1,2)$	$H(1,2)h^2$	$H(1,2)h$	$H(1,2)h^3$
$T(1,2)H$	$T(1,2)$	$T(3,4)$	$T(1,3,4)$	$T(1,2,3)$
$Th(1,2)H$	$T(1,4,3)$	$T(1,2,4)$	$T(2,3)$	$T(1,4)$

FIGURE 3. The top diagram shows the double coset $H(1,2)H$ when $G = \mathrm{Sym}_4$ and $H = \langle(1,2,3,4)\rangle$, and its action on the six faces of a die. Rows are left cosets and columns are right cosets. For instance, the permutation $(1,2)$ swaps two diagonals of the cube, and corresponds to the 180° rotation about the axis through the centre of the edge between faces ⚀ and ⚁ and the centre of the edge between faces ⚄ and ⚅, resulting in a permutation (⚀,⚁)(⚂,⚃)(⚄,⚅) of the faces. Shaded regions indicate the double cosets TxT where $T = \langle(1,2)(3,4)\rangle$. The division into right cosets of T relevant to Example 13.7 is shown in the lower diagram. For instance $T(1,2)T = T(1,2) \cup T(3,4)$ contains the four permutations in the white region.

We now use Corollary 13.3 to compute the space perpendicular to $\Theta(e) \cap \mathbb{C}[HxH]$ for every $e \in E^\bullet(H)$ and every double coset HxH.

13.2.4. *Strong lumping.* Fix the real idempotent $1 \in E^\bullet(H)$. The algebra $\Theta(1)$ is the strong lumping algebra (see Proposition 5.15). That is, $w \in \Theta(1)$ if and only if the left-invariant random walk driven by w lumps weakly for all initial distributions. By Corollary 13.3, this holds for $w \in \mathbb{C}[HxH]$ if and only if

$$x \in \langle \xi x \eta, \varsigma x \eta, \bar{\xi} x \eta \rangle^\perp.$$

(Note the first summand in this corollary vanishes.) If $x = \mathrm{id}_H$ or $x = (1,3)$ then since $Hx = xH$ in these cases, by Corollary 13.4, there is no constraint from this double coset. Hence

$$\Theta(1)^\perp = \big\langle \xi(1,2)\eta, \varsigma(1,2)\eta, \bar{\xi}(1,2)\eta \big\rangle$$

and there are just 3 linear constraints.

13.2.5. *Exact lumping.* Fix the real idempotent $\eta \in E^\bullet(H)$. The algebra $\Theta(\eta)$ is the exact lumping algebra $\Theta(\eta)$ seen in Proposition 5.16. (See Definition 2.19 for the definition of exact lumping.) By Corollary 13.3, the weight $w \in \mathbb{C}[HxH]$ lumps exactly if and only if

$$x \in \langle \eta x \xi, \eta x \varsigma, \eta x \bar{\xi} \rangle^\perp.$$

(Note the second summand in this corollary vanishes.) Again if $x = \mathrm{id}_H$ or $x = (1,3)$ then since $Hx = xH$ in these cases, by Corollary 13.4, there is no constraint from this double coset. Hence

$$\Theta(\eta_H)^\perp = \big\langle \eta(1,2)\xi, \eta(1,2)\varsigma, \eta(1,2)\bar{\xi} \big\rangle^\perp$$

and again there are 3 linear constraints. This should be expected from Theorem 1.9, which, informally stated, says that the exact lumping algebra $\Theta(\eta_H)$ is dual to the strong lumping algebra $\Theta(1)$.

13.2.6. *The first weak lumping algebra.* Fix $P = \{\eta, \xi, \bar{\xi}\} \subseteq \widehat{H}$ and the real idempotent $e_P = \eta + \xi + \bar{\xi} \in E^\bullet(H)$. Similar arguments to the strong and exact cases using Corollaries 13.3 and 13.4 show that

$$\Theta(e_P)^\perp = \big\langle \eta(1,2)\varsigma, \xi(1,2)\varsigma, \bar{\xi}(1,2)\varsigma, \xi(1,2)\eta, \bar{\xi}(1,2)\eta \big\rangle. \tag{13.3}$$

In particular, there is no constraint from the double cosets H and $H(1,3)H$ and the dimension of $\Theta(e_P)$ is $24 - 5 = 19$. It is instructive to verify this using the dimension formula of Corollary 7.3. The four characters of C_4 are $\mathbb{1}, \gamma, \mathrm{sgn}, \bar{\gamma}$, corresponding to the four idempotents $\eta, \xi, \varsigma, \bar{\xi}$, and

$$\begin{aligned} \mathbb{1}\big\uparrow_H^G &= \chi^{(4)} + \chi^{(2,2)} + \chi^{(2,1,1)} \\ \bar{\gamma}\big\uparrow_H^G = \gamma\big\uparrow_H^G &= \chi^{(3,1)} + \chi^{(2,1,1)} \\ \mathrm{sgn}\big\uparrow_H^G &= \chi^{(3,1)} + \chi^{(2,2)} + \chi^{(1,1,1,1)}, \end{aligned}$$

and thus the relevant coefficients are as shown in the table below.

ψ	$\chi^{(4)}$	$\chi^{(3,1)}$	$\chi^{(2,2)}$	$\chi^{(2,1,1)}$	$\chi^{(1,1,1,1)}$
$a_\psi = \langle (\gamma + \bar{\gamma})\uparrow_H^G, \psi \rangle$	0	2	0	2	0
$c_\psi = \langle \mathbb{1}\uparrow_H^G, \psi \rangle$	1	0	1	1	0
$d_\psi = \psi(1)$	1	3	2	3	1

The dimension formula therefore gives

$$\begin{aligned} \dim \Theta(e_P) &= \sum_{\psi \in \operatorname{Irr} G} (a_\psi^2 + a_\psi c_\psi + c_\psi^2 - a_\psi d_\psi - c_\psi d_\psi + d_\psi^2). \\ &= 1 + 7 + 3 + 7 + 1 = 19 \end{aligned}$$

as expected. To make (13.3) explicit, we represent a weight

$$w = \sum_{i=0}^{4} \sum_{j=0}^{4} w\big(h^i(1,2)h^j\big) \cdot h^i(1,2)h^j$$

supported on $H(1,2)H$ as a 4×4 matrix, in which $w\big(h^i(1,2)h^j\big)$ is in the row $h^i(1,2)H$ and column $H(1,2)h^j$, when i and j are ordered $0, 2, 1, 3$ as in Figure 3. Using this notation, the algebra $\Theta(e_P)$ is then the space of all weights whose coefficients on $H(1,2)H$ are orthogonal to the five matrices

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ i & i & -i & -i \\ -i & -i & i & i \end{pmatrix}, \begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -i & -i & i & i \\ i & i & -i & -i \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ i & i & i & i \\ -i & -i & -i & -i \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ -i & -i & -i & -i \\ i & i & i & i \end{pmatrix}$$

or equivalently, by taking obvious linear combinations, to the five matrices

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}.$$

We use this description of $\Theta(e_P)$ to illustrate Theorem 1.11. The orbital matrices (see §9.1) for the three double cosets H, $H(1,3)$ and $H(1,2)H$ are

$$\begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix}, \quad \begin{pmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix}, \quad \begin{pmatrix} \cdot & \cdot & 1 & 1 & 1 & 1 \\ \cdot & \cdot & 1 & 1 & 1 & 1 \\ 1 & 1 & \cdot & \cdot & 1 & 1 \\ 1 & 1 & \cdot & \cdot & 1 & 1 \\ 1 & 1 & 1 & 1 & \cdot & \cdot \\ 1 & 1 & 1 & 1 & \cdot & \cdot \end{pmatrix}$$

where the left cosets of H appear in the order H, $(1,3)H$, $(1,2)H$, $h^2(1,2)H$, $h(1,2)H$, $h^3(1,2)H$, and for readability $\cdot$ denotes a 0 entry. For example, the weight defined by the third matrix is simply $\eta(1,2)\eta$, giving equal weight to all elements in the double coset $H(1,2)H$, and so corresponding to the all-ones 4×4 matrix in the notation above. Clearly it is orthogonal to all five matrices. A similar argument for the other two cosets shows that, as expected, weights in the Hecke algebra $\eta\mathbb{C}[\mathrm{Sym}_4]\eta$ satisfy (13.3) and so lump stably, in the sense of Definition 1.3 for the ideal $\mathbb{C}[G]e_P$. Of course such weights also lump strongly and exactly, by Theorem 1.11.

13.2.7. *A weak lumping weight that does not lump strongly or exactly, from the first weak lumping algebra.* From the final 5 matrices above, we see that $w \in \Theta(e_P)$ if and only if w satisfies the five constraints

$$\begin{aligned} w\bigl(H(1,2)\bigr) + w\bigl(H(1,2)h^2\bigr) &= w\bigl(H(1,2)h\bigr) + w\bigl(H(1,2)h^3\bigr) \\ w\bigl((1,2)H\bigr) &= w\bigl(h^2(1,2)H\bigr) \\ w\bigl(h(1,2)H\bigr) &= w\bigl(h^3(1,2)H\bigr) \\ w\bigl((1,2)\bigr) + w\bigl((1,2)h^2\bigr) &= w\bigl(h^2(1,2)\bigr) + w\bigl(h^2(1,2)h^2\bigr) \\ w\bigl(h(1,2)h\bigr) + w\bigl(h(1,2)h^3\bigr) &= w\bigl(h^3(1,2)h\bigr) + w\bigl(h^3(1,2)h^3\bigr). \end{aligned} \tag{13.4}$$

It is notable that it is not obvious that these conditions even define a subalgebra of $\mathbb{C}[G]$. We now use this description to give, in the same spirit as the example in §1.2.3, a weight which lumps weakly to the left cosets G/H but not strongly or exactly.

Example 13.5. Let w be the weight supported on the double coset $H(1,2)H$ as written below in our usual convention so that the entry $h^i(1,2)h^j$ is in the position indicated by Figure 3, using the order $0, 2, 1, 3$.

$$\frac{1}{12}\begin{pmatrix} 2 & 1 & 1 & 2 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Explicitly,

$$\begin{aligned} w &= \frac{1}{12}\bigl(2(1,2) + (1,2)h^2 + (1,2)h + 2(1,2)h^3 + 3h^2(1,2)h^2 + 3h^2(1,2)h\bigr) \\ &= \frac{1}{12}\bigl(2(1,2) + (1,4,2,3) + (1,3,4) + 2(2,4,3) + 3(3,4) + 3(1,4,2)\bigr). \end{aligned}$$

This matrix is orthogonal to either set of five matrices shown above, and so the weight w lumps weakly. By Corollary 1.10, it does not lump strongly or exactly because the matrix has neither constant row sums, nor constant column sums.

Remark 13.6. If $L = \mathbb{C}[G]e_P$ then $L^\circ = \mathbb{C}[G](e_P - \eta_H)$ and, by Proposition 13.2, a necessary and sufficient condition for $w \in \mathrm{RId}(L^\circ)$ is that w satisfies the final four equations in (13.4); now Proposition 12.10 implies that a weight w satisfying these equations is weakly lumping if and only if there exists $v \in L^\circ$ such that $w\bigl(H(1,2)h^j\bigr) + v\bigl(H(1,2)h^j\bigr)$ is constant as j varies. Since v must be real and $e_P - \eta_H = \xi + \bar{\xi} = \frac{1}{2}(1 - h + h^2 - h^3)$ an equivalent condition is that

$$w\bigl(H(1,2)h^j\bigr) + v\bigl(H(1,2)h^j\bigr) \tag{13.5}$$

is constant as j varies, for some $v \in \langle h^i(1,2)(1 - h + h^2 - h^3) : 0 \le i \le 3\rangle$. Since any such v satisfies $v\bigl(H(1,2)\bigr) = v(H(1,2)h^2$ and $v\bigl(H(1,2)h\bigr) = v\bigl(H(1,2)h^3\bigr)$, whenever the final four equations in (13.4) hold, the first holds if and only if condition (13.5) holds. This verifies the conclusion of Proposition 12.10 for the first weak lumping algebra.

13.2.8. *The second weak lumping algebra.* The final real idempotent $e_P \in E^\bullet(H)$ is defined by taking $P = \{\eta, \varsigma\} \subseteq \widehat{H}$. This corresponds to the time reversal of the previous example, so we just give the result of Corollaries 13.3 and 13.4 that

$$\Theta(e_P)^\perp = \big\langle \varsigma(1,2)\xi, \varsigma(1,2)\xi, \varsigma(1,2)\bar{\xi}, \eta(1,2)\xi, \eta(1,2)\bar{\xi} \big\rangle.$$

We finish by giving an alternative description of $\Theta(e_P)^\perp$ using Proposition 11.1. Let $T = \langle (1,3)(2,4) \rangle \cong C_2$. We have $T \leq H \leq G$, and $\eta_T = \eta + \varsigma = e_P$. Hence $\mathbb{C}[G]e_P$ is a weak lumping Gurvits–Ledoux ideal (in the sense of Definition 5.11) for the weight w if and only if

(a′) $w(Tg)$ is constant for $Tg \subseteq TxT$ for all $TxT \in T\backslash G/T$, and
(b′) $w(TgH)$ is constant for $TgH \subseteq HxH$ for all $HxH \in H\backslash G/H$.

Many choices of g and x render $Tg = TxT$ or $TgH = HxH$, and thus impose no constraint on w. We give below the remaining five equations which define $\Theta(e_P)$:

(a′) $w\big(T(1,2)\big) = w\big(T(3,4)\big)$,
$w\big(T(1,3,4)\big) = w\big(T(1,2,3)\big)$,
$w\big(T(1,2,4)\big) = w\big(T(1,4,3)\big)$,
$w\big(T(1,4)\big) = w\big(T(2,3)\big)$;
(b′) $w\big(T(1,2)H\big) = w\big(T(1,4)H\big)$.

These equations specify that the weights of each right coset in the shaded regions of Figure 3 are equal. For instance $T(1,2) = \{(1,2),(1,3,2,4)\}$ and $T(3,4) = \{(3,4),(1,4,3,2)\}$ together form the top-left region in this figure.

Example 13.7. In Example 13.5 we saw that the weight

$$w = \frac{1}{12}\big(2(1,2) + (1,4,2,3) + (1,3,4) + 2(2,4,3) + 3(3,4) + 3(1,4,2)\big)$$

lumps weakly to G/H with stable ideal $\mathbb{C}[G](\eta + \xi + \bar{\xi})$. Noting that $1 - (\eta + \xi + \bar{\xi}) + \eta = \eta + \varsigma$, by Theorem 1.9, the weight

$$w^\star = \frac{1}{12}\big(2(1,2) + (1,3,2,4) + (1,4,3) + 2(2,3,4) + 3(3,4) + 3(1,2,4)\big)$$

lumps weakly to G/H with the stable ideal $\mathbb{C}[G](\eta + \varsigma)$ relevant to the final idempotent e_P in this subsection. This may be checked directly using Figure 3, noting that $w^\star$ is supported on the 8 permutations in the left half of the diagram, and the weights of the two right cosets of T in the top-left are equal, and similarly for the bottom-left. We saw in Example 13.5 that w does not lump strongly or exactly and so, by the $\star$-duality in Theorem 1.9, neither does $w^\star$.

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