NOTES ON THE WEYL CHARACTER FORMULA

The aim of these notes is to give a self-contained algebraic proof of the Weyl Character Formula. The necessary background results on modules for $\mathsf{sl}_2(\mathbf{C})$ and complex semisimple Lie algebras are outlined in the first two sections. Some technical details are left to the exercises at the end; solutions are provided when the exercise is needed for the proof.

1. Representations of $\mathsf{sl}_2(\mathbf{C})$

Define

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and note that $\langle h, e, f \rangle = \mathsf{sl}_2(\mathbf{C})$. Let u, v be the canonical basis of $E = \mathbf{C}^2$. Then each $\operatorname{Sym}^d E$ is irreducible with u^d spanning the highest-weight space of weight d and, up to isomorphism, $\operatorname{Sym}^d E$ is the unique irreducible $\mathsf{sl}_2(\mathbf{C})$-module with highest weight d. (See Exercises 1.1 and 1.2.) The diagram below shows the actions of h, e and f on the canonical basis of $\operatorname{Sym}^d E$: loops show the action of h, arrows to the right show the action of e, arrow to the left show the action of f.

$$\begin{tikzcd}
\cdots & u^{d-c-1}v^{c+1} \arrow[l, bend left, "c-1"] \arrow[loop above, "d-2c-2"] \arrow[r, bend left, "d-c+1"] & u^{d-c}v^{c} \arrow[l, bend left, "c"] \arrow[loop above, "d-2c"] \arrow[r, bend left, "d-c"] & u^{d-c+1}v^{c-1} \arrow[l, bend left, "c+1"] \arrow[loop above, "d-2c+2"] \arrow[r, bend left, "d-c-1"] & \cdots
\end{tikzcd}$$

In particular

(a) the eigenvalues of h on $\operatorname{Sym}^d E$ are $-d, -d+2, \ldots, d-2, d$ and each h-eigenspace is 1-dimensional,

(b) if $w \in \operatorname{Sym}^d E$ and $h \cdot w = (d-2c)w$ then $f \cdot e \cdot w = c(d-c+1)w$.

If V is an arbitrary $\mathsf{sl}_2(\mathbf{C})$-module then, by Weyl's Theorem (see [1, Appendix B] or [3, §6.3]), V decomposes as a direct sum of irreducible $\mathsf{sl}_2(\mathbf{C})$-submodules. Let $V_r = \{v \in V : h \cdot w = rv\}$ for $r \in \mathbf{Z}$. Then (a) implies

(c) if $r \geq 0$ then the number of irreducible summands of V with highest weight r is $\dim V_r - \dim V_{r+2}$.

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2. Prerequisites on complex semisimple Lie algebras

In this section we recall the basic setup of a Cartan subalgebra H inside a complex semisimple Lie algebra L, a lattice of weights $\Lambda \subseteq H^\star_{\mathbf{R}}$ and a root system $\Phi \subseteq \Lambda$. The mathematically most interesting parts are that H is self-centralizing (see Exercise 2.2) and the trick used to construct an $\mathsf{sl}_2(\mathbf{C})$-subalgebra corresponding to each root. For an example of all the theory below, see Exercise 2.7.

Cartan subalgebras. We define a *Cartan subalgebra* of L to be a Lie subalgebra H of L maximal subject to the condition that $\operatorname{ad} h : L \to L$ is diagonalizable for all $h \in H$. It is an interesting fact (see Exercise 2.1) that any Cartan subalgebra is abelian. We may therefore decompose L as a direct sum of simultaneous eigenspaces for the elements of H. To each simultaneous eigenspace V we associate the unique $\alpha \in H^\star$ such that $(\operatorname{ad} h)x = \alpha(h)x$ for all $h \in H$ and $x \in V$. For $\alpha \in H^\star$ let

$$L_\alpha = \{x \in L : (\operatorname{ad} h)x = \alpha(h)x \text{ for all } h \in H,\, x \in V\}$$

and let Φ be the set of all non-zero $\alpha \in H^\star$ such that $L_\alpha \neq 0$. The elements of Φ are called *roots* and L_α is the *root space* corresponding to $\alpha \in \Phi$ and we have

$$L = L_0 \oplus \Big(\bigoplus_{\alpha \in \Phi} L_\alpha\Big).$$

Note that L_0 is the centralizer of H in L. It is an important and non-obvious fact (see Exercise 2.2) that $L_0 = H$, so H is self-centralizing: An easy calculation shows that

$$[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta} \quad \text{for all } \alpha, \beta \in \Phi_0. \tag{1}$$

Killing form. The *Killing form* on L is the bilinear form $\kappa(x, y) = \operatorname{Tr}(\operatorname{ad} x \circ \operatorname{ad} y)$. By Cartan's Criterion κ is non-degenerate. It follows from (1) that if $x \in L_\alpha$ and $y \in L_\beta$ where $\alpha, \beta \in \Phi_0$, then $\operatorname{ad} x \circ \operatorname{ad} y$ is nilpotent, unless $\alpha + \beta = 0$. Therefore if $\alpha, \beta \in \Phi_0$ then $L_\alpha \perp L_\beta$ unless $\beta = -\alpha$. Hence α is a root if and only if $-\alpha$ is a root and the restriction of κ to $L_\alpha \times L_{-\alpha}$ is non-degenerate. In particular, the restriction of κ to $H \times H$ is non-degenerate. For each $\alpha \in \Phi$, let $t_\alpha \in H$ be the unique element of H such that

$$\kappa(t_\alpha, h) = \alpha(h) \quad \text{for all } h \in H.$$

sl_2 *subalgebras.* Choose $e \in L_\alpha$ and $f \in L_{-\alpha}$ such that $\kappa(e, f) \neq 0$. By the associativity of the Killing form

$$\kappa(h, [e, f]) = \kappa([h, e], f) = \alpha(h)\kappa(e, f) \quad \text{for all } h \in H.$$

Since κ is non-degenerate on H, there exists $h \in H$ such that $\alpha(h) = \kappa(t_\alpha, h) \neq 0$. Since $\kappa(e, f) \neq 0$, the previous equation then implies that $[e, f] \neq 0$. Consider the Lie subalgebra

$$\langle e, f, [e, f] \rangle$$

of L. Since $[e, f] \in [L_\alpha, L_{-\alpha}] \subseteq H$ we have $\big[[e, f], e\big] = \alpha([e, f])e$ and $\big[[e, f], f\big] = -\alpha([e, f])f$.

If $\alpha([e, f]) = 0$ then $[e, f]$ is central in $\langle e, f, [e, f] \rangle$. By Exercise 2.3 below $[e, f]$ is nilpotent. But $[e, f] \in H$ and all the elements of H are semisimple. So $[e, f] = 0$, which contradicts the previous paragraph. Therefore $\alpha([e, f]) \neq 0$ and we can scale e so that $\alpha([e, f]) = 2$ and so $\langle e, f, [e, f] \rangle \cong \mathsf{sl}_2(\mathbf{C})$.

For each $\alpha \in \Phi$ let $\langle e_\alpha, f_\alpha, h_\alpha \rangle$ be a subalgebra of L constructed as above so that

$$[e_\alpha, f_\alpha] = h_\alpha, \quad [h_\alpha, e_\alpha] = 2e_\alpha, \quad [h_\alpha, f_\alpha] = 2f_\alpha.$$

We may suppose that these elements are chosen so that $e_{-\alpha} = f_\alpha$ and $f_{-\alpha} = e_\alpha$ for each $\alpha \in \Phi$.

Relationship between t_α and h_α. By choice of t_α we have $\kappa(t_\alpha, h) = \alpha(h)$ for all $h \in H$. By associativity of the Killing form we also have

$$\kappa([e_\alpha, f_\alpha], h) = \kappa(e_\alpha, [f_\alpha, h]) = \kappa(e_\alpha, \alpha(h) f_\alpha) = \alpha(h)\kappa(e_\alpha, f_\alpha).$$

Hence

$$\kappa\Big(t_\alpha - \frac{[e_\alpha, f_\alpha]}{\kappa(e_\alpha, f_\alpha)}, h\Big) = 0 \quad \text{for all } h \in H.$$

Since the restriction of κ to $H \times H$ is non-degenerate it follows that

$$t_\alpha = \frac{h_\alpha}{\kappa(e_\alpha, f_\alpha)}. \tag{2}$$

Since $\kappa(t_\alpha, t_\alpha) = \alpha(t_\alpha)$, this implies the useful relations

$$2 = \alpha(h_\alpha) = \kappa(t_\alpha, h_\alpha) = \frac{\kappa(h_\alpha, h_\alpha)}{\kappa(e_\alpha, f_\alpha)} = \kappa(e_\alpha, f_\alpha)\kappa(t_\alpha, t_\alpha). \tag{3}$$

Transport of the Killing form to $H^\star_{\mathbf{R}}$. We saw earlier that for all $\alpha \in \Phi$ there exists $h \in H$ such that $\alpha(h) \neq 0$. It follows that Φ spans $H^\star$ and there is a unique bilinear form $(\ , \ )$ on $H^\star$ such that

$$(\alpha, \beta) = \kappa(t_\alpha, t_\beta) \quad \text{for } \alpha, \beta \in \Phi.$$

By (2) and (3) we have the fundamental formula

$$\frac{2(\alpha, \beta)}{(\beta, \beta)} = \kappa\Big(t_\alpha, \frac{2t_\beta}{\kappa(t_\beta, t_\beta)}\Big) = \kappa(t_\alpha, h_\beta) = \alpha(h_\beta). \tag{4}$$

Note also that $\alpha(h_\beta)$ is an eigenvalue of h_β in the finite-dimensional $\mathsf{sl}(\beta)$-module L. It follows that $(\ , \ )$ takes real values on the roots and from the equation $\kappa(h,k) = \sum_{\alpha\in\Phi} \alpha(h)\alpha(k)$ for $h,k \in H$, we see that it is a real-valued inner-product on $H^\star_{\mathbf{R}} = \langle \alpha : \alpha \in \Phi\rangle_{\mathbf{R}}$. Exercise 2.4 shows that the angles between the roots are determined by (4). (In fact if L is a simple Lie algebra then Φ is a connected root system and $(\ ,\ )$ is completely determined by (4) and (α,α) for any single root $\alpha \in \Phi$.)

Angled brackets notation. It will be convenient to define

$$\langle \lambda, \mu \rangle = \frac{2(\lambda,\mu)}{(\mu,\mu)}$$

for $\lambda, \mu \in H^\star_{\mathbf{R}}$. Note that the form $\langle\ ,\ \rangle$ is linear only in its first component. This notation will often be used when $\mu \in \Phi$, in which case (4) implies that $\langle \lambda, \beta\rangle = \lambda(h_\beta)$.

Fundamental dominant weights. Recall that $\{\alpha_1,\ldots,\alpha_\ell\}$ is a base for Φ if element of Φ can be written uniquely as either a non-negative or non-positive integral linear combination of the α_i. (For a proof that every root system has a basis, see [1, Theorem 11.10] or [3, Theorem 10.1].) Fix, once and for all, a base $\{\alpha_1,\ldots,\alpha_\ell\}$ for Φ and let Φ^+ be the set of positive roots with respect to this basis. There exist unique $\omega_1,\ldots,\omega_\ell \in H^\star$ such that, for all $i,j \in \{1,\ldots,\ell\}$,

$$\omega_i(h_{\alpha_j}) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $\Lambda = \langle \omega_1,\ldots,\omega_\ell\rangle_{\mathbf{Z}} \subseteq H^\star$.

Weight space decomposition. The elements of H act semisimply in any finite-dimensional L-module (see [3, Corollary 6.3]). By Section 1, the eigenvalues of each h_{α_j} are integral. Hence if V is a finite-dimensional L-module then

$$V\!\downarrow_H = \bigoplus_{\lambda\in\Lambda} V_\lambda$$

where

$$V_\lambda = \{v \in V : h\cdot v = \lambda(h)v \text{ for all } h \in H\}.$$

(The root spaces defined earlier are weight spaces for the action of L on itself by the adjoint representation.) We shall say that an element of V lying in some non-zero V_λ is a *weight vector*. Starting with any weight vector, and then repeatedly applying the elements e_α for $\alpha \in \Phi^+$, it follows that V contains a weight vector v such that $e_\alpha \cdot v = 0$ for all $\alpha \in \Phi^+$. Such a vector is said to be a *highest-weight vector* with respect to the base $\{\alpha_1,\ldots,\alpha_\ell\}$. By Exercise 2.6, the submodule of V generated by a highest weight vector is irreducible.

3. Freudenthal's Formula

Let V be an irreducible L-module of highest weight $\mu \in \Lambda$. Let $n_\nu = \dim V_\nu$ for each $\nu \in \Lambda$. The aim of this section is to prove *Freudenthal's Formula*, that if $\lambda \in \Lambda$ then

$$\big(||\mu+\delta||^2 - ||\lambda+\delta||^2\big) n_\lambda = 2 \sum_{\alpha\in\Phi^+} \sum_{m=1}^{\infty} n_{\lambda+m\alpha}(\lambda+m\alpha,\alpha)$$

where $\delta = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. The key idea in this proof (which is based on [5, VIII.2]) is to calculate the scalar by which a central element in the universal enveloping algebra $\mathcal{U}(L)$ acts on V, using the theory of $\mathsf{sl}_2(\mathbf{C})$-modules in Section 1. The following lemma gives a construction of such central elements.

Lemma 3.1. *Suppose that $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ are bases of L such that*

$$\kappa(x_i,y_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j. \end{cases}$$

Then $\sum_{i=1}^n x_i y_i$ is in the centre of $\mathcal{U}(L)$.

Proof. See Exercise 3.3. □

Let $\alpha,\beta\in\Phi$. By (1) we have $\kappa(e_\alpha,f_\beta)=0$ whenever $\alpha\neq\beta$ and by (3) we have $\kappa(e_\alpha,f_\alpha) = 2/\kappa(t_\alpha,t_\alpha) = 2/(\alpha,\alpha)$ and $\kappa(t_\alpha,h_\alpha)=2$ for all $\alpha\in\Phi$. Lemma 3.1 therefore implies that

$$\Gamma = \sum_{\alpha\in\Phi} \frac{(\alpha,\alpha)}{2} f_\alpha e_\alpha + \frac{1}{2}\sum_{j=1}^{\ell} t_{\alpha_j} h_{\alpha_j}$$

is in the centre of $\mathcal{U}(L)$. We may assume that if $\alpha\in\Phi^+$ then $e_{-\alpha}=f_\alpha$ and $f_{-\alpha}=e_\alpha$. Hence $f_{-\alpha}e_{-\alpha} = e_\alpha f_\alpha = h_\alpha + f_\alpha e_\alpha$ and

$$\Gamma = \sum_{\alpha\in\Phi^+}\frac{(\alpha,\alpha)}{2}h_\alpha + \sum_{\alpha\in\Phi^+}(\alpha,\alpha)f_\alpha e_\alpha + \frac{1}{2}\sum_{j=1}^{\ell} t_{\alpha_j}h_{\alpha_j}.$$

The action of each of the three summands of Γ preserves the weight spaces V_λ. The next three lemmas determine the traces of these summands on each V_λ. The first explains the appearance of δ in Freudenthal's Formula.

Lemma 3.2. *If $\lambda\in\Lambda$ and $v\in V_\lambda$ then*

$$\sum_{\alpha\in\Phi^+}\frac{(\alpha,\alpha)}{2}h_\alpha\cdot v = (\lambda,2\delta)v.$$

Proof. Using (4) we get

$$\sum_{\alpha\in\Phi^+}\frac{(\alpha,\alpha)}{2}\lambda(h_\alpha) = \sum_{\alpha\in\Phi^+}\frac{(\alpha,\alpha)}{2}\frac{2(\lambda,\alpha)}{(\alpha,\alpha)} = \sum_{\alpha\in\Phi^+}(\lambda,\alpha) = (\lambda,2\delta)$$

as required. □

Lemma 3.3. *If $\alpha \in \Phi$ and $\lambda \in \Lambda$ then*

$$(\alpha, \alpha) \operatorname{Tr}_{V_\lambda}(f_\alpha e_\alpha) = 2 \sum_{m=1}^{\infty} n_{\lambda+m\alpha}(\lambda + m\alpha, \alpha).$$

Proof. Since $\frac{2(\lambda+m\alpha,\alpha)}{(\alpha,\alpha)} = \langle \lambda + m\alpha, \alpha \rangle$, it is equivalent to prove that

$$\operatorname{Tr}_{V_\lambda}(f_\alpha e_\alpha) = \sum_{m=1}^{\infty} n_{\lambda+m\alpha} \langle \lambda + m\alpha, \alpha \rangle.$$

Let $W = \bigoplus_{c \in \mathbf{Z}} V_{\lambda + c\alpha}$. Note that W is a direct sum of weight spaces for the action of H, and that W is an $\mathsf{sl}(\alpha)$-submodule of V. We may write

$$W = U^{(1)} \oplus \cdots \oplus U^{(d)}$$

where each $U^{(i)}$ is an irreducible $\mathsf{sl}(\alpha)$-module.

Assume first of all that $\lambda(h_\alpha) \geq 0$. Suppose that $U^{(i)}_\lambda \neq 0$. Choose m maximal such that $U^{(i)}_{\lambda+m\alpha} \neq 0$. Then $U^{(i)}$ has highest weight $(\lambda + m\alpha)(h_\alpha)$ as an $\mathsf{sl}(\alpha)$-module and by (b) in Section 1, the scalar by which $f_\alpha e_\alpha$ acts on a vector in $U^{(i)}_\lambda$ is

$$m\big((\lambda + m\alpha)(h_\alpha) - m + 1\big) = m\big(\lambda(h_\alpha) + m + 1\big).$$

It follows from (c) in Section 1 that the number of summands $U^{(i)}$ with highest weight $(\lambda + m\alpha)(h_\alpha)$ as an $\mathsf{sl}(\alpha)$-module is $n_{\lambda+m\alpha} - n_{\lambda+(m+1)\alpha}$. Hence

$$\begin{aligned}
\operatorname{Tr}_{V_\lambda}(f_\alpha e_\alpha) &= \sum_{m=0}^{\infty} \big(n_{\lambda+m\alpha} - n_{\lambda+(m+1)\alpha}\big) m(\lambda(h_\alpha) + m + 1) \\
&= \sum_{m=1}^{\infty} n_{\lambda+m\alpha}\big(m(\lambda(h_\alpha) + m + 1) - (m-1)(\lambda(h_\alpha) + m)\big) \\
&= \sum_{m=1}^{\infty} n_{\lambda+m\alpha}(\lambda(h_\alpha) + 2m).
\end{aligned}$$

as required. Note that this equation holds even when $V_\lambda = 0$, since the argument just given shows that both sides are zero.

If $\lambda(h_\alpha) \leq 0$ then a similar calculation (see Exercise 3.4) shows that $f_\alpha e_\alpha$ acts as the scalar $-\sum_{b=0}^{\infty} n_{\lambda-b\alpha} \langle \lambda - b\alpha, \alpha \rangle$ on V_λ. Now $\sum_{c=-\infty}^{\infty} n_{\lambda+c\alpha} \langle \lambda + c\alpha, \alpha \rangle = 0$ since each irreducible summand $U^{(i)}$ contributes the sum of the h_α eigenvalues on $U^{(i)}$, which is 0 by (a) in Section 1. Adding these two equations we get the required formula. □

Lemma 3.4. *Let $\lambda \in \Lambda$. If $v \in V_\lambda$ then*

$$\frac{1}{2} \sum_{j=1}^{\ell} t_{\alpha_j} h_{\alpha_j} \cdot v = (\lambda, \lambda) v$$

Proof. We saw earlier that $\frac{1}{2}t_{\alpha_1}, \ldots, \frac{1}{2}t_{\alpha_\ell}$ and $h_{\alpha_1}, \ldots, h_{\alpha_\ell}$ are dual bases of $H^\star$ with respect to the Killing form κ on $H \times H$. By Exercise 3.2(ii)

$$\frac{1}{2}\sum_{i=1}^{\ell} \lambda(t_{\alpha_j})\lambda(h_{\alpha_j}) = (\lambda, \lambda)$$

as required. □

Since Γ is central in $\mathcal{U}(L)$ it acts as a scalar on V, say γ. Let $\lambda \in \Lambda$. By Lemmas 3.2, 3.3 and 3.4, we have

$$n_\lambda \gamma = \mathrm{Tr}_{V_\lambda}(f_\alpha e_\alpha) = (\lambda, 2\delta)n_\lambda + 2\sum_{\alpha \in \Phi^+}\sum_{m=1}^{\infty} n_{\lambda+m\alpha}(\lambda + m\alpha, \alpha) + (\lambda, \lambda)n_\lambda.$$

Recall that V has highest weight μ. Since $e_\alpha \cdot V_\mu = 0$ for all $\alpha \in \Phi^+$, $n_\mu = 1$, and $(\lambda, 2\delta) + (\lambda, \lambda) = ||\lambda + \delta||^2 - ||\delta||^2$, the previous equation implies

$$\gamma = ||\mu + \delta||^2 - ||\delta^2||.$$

Comparing these two equations we obtain

$$\big(||\mu + \delta||^2 - ||\lambda + \delta||^2\big)n_\lambda = 2\sum_{\alpha \in \Phi^+}\sum_{m=1}^{\infty} n_{\lambda+m\alpha}(\lambda + m\alpha, \alpha)$$

as stated in Freudenthal's Formula. For an immediate application of Freudenthal's Formula see Exercise 3.5 below.

4. Statement of Weyl Character Formula

Formal exponentials and characters. For each $\lambda \in \Lambda$ we introduce a formal symbol $e(\lambda)$ which we call the *formal exponential of* λ. Let $\mathbf{Q}[\Lambda]$ denote the abelian group with $\mathbf{Z}$-basis $\{e(\lambda) : \lambda \in \Lambda\}$. We make $\mathbf{Q}[\Lambda]$ into an algebra by defining the multiplication on basis elements by

$$\mathrm{e}(\lambda)e(\lambda') = \mathrm{e}(\lambda + \lambda') \quad \text{for } \lambda, \lambda' \in \Lambda.$$

Note that $\mathrm{e}(0) = 1$ and that each $\mathrm{e}(\lambda)$ is invertible, with inverse $\mathrm{e}(-\lambda)$. This definition is motivated by 1-parameter subgroups: see Exercise 4.1. Given an L-module V, we define the *formal character* of L by

$$\chi_V = \sum_{\lambda \in \Lambda} (\dim V_\lambda)e(\lambda) \in \mathbf{Q}[\Lambda].$$

Weyl group. Let $S_\beta : H^\star_{\mathbf{R}} \to H^\star_{\mathbf{R}}$ denote the reflection corresponding to $\beta \in \Phi$ as defined by

$$S_\beta(\theta) = \theta - \frac{2(\theta, \beta)}{(\beta, \beta)}\beta \quad \text{for } \theta \in H^\star_{\mathbf{R}}.$$

The alterative forms $S_\beta(\theta) = \theta - \langle \theta, \beta \rangle \beta = w - \theta(h_\beta)\alpha$ are often useful. By definition the *Weyl group* of L is the group generated by the S_β for $\beta \in \Phi$. We define $\varepsilon(w) = 1$ if w is a product of an even number of reflections, and

$\varepsilon(w) = -1$ otherwise. The Weyl group W acts on $\mathbf{Q}[\Lambda]$ by $w \cdot \mathrm{e}(\lambda) = e(w(\lambda))$ for $w \in W$ and $\lambda \in \Lambda$.

Symmetric and antisymmetric elements. We say that an element $f \in \mathbf{Q}[\Lambda]$ is *symmetric* if $w \cdot f = f$ for all $w \in W$ and *antisymmetric* if $w \cdot f = \varepsilon(w) f$ for all $w \in W$. By Exercise 4.3(iv), $f \in \mathbf{Q}[\Lambda]$ is antisymmetric if and only if

$$f = g \sum_{w \in W} \varepsilon(w)\, w \cdot \mathrm{e}(\delta)$$

for some symmetric g.

Weyl Character Formula. We may now state the main result. By the result on antisymmetric elements of $\mathbf{Q}[\Lambda]$ just mentioned, the right-hand side in the formula below is a well-defined symmetric element of $\mathbf{Q}[\Lambda]$.

Theorem 4.1 (Weyl Character Formula)**.** *Let V be the irreducible L-module of highest weight $\mu \in \Lambda$. Then*

$$\chi_V = \frac{\sum_{w \in W} \varepsilon(w)\, w \cdot \mathrm{e}(\mu + \delta)}{\sum_{w \in W} \varepsilon(w)\, w \cdot \mathrm{e}(\delta)}.$$

Some applications of the Weyl Character Formula are given in Exercises 4.4, 4.5 and 4.6. Kostant's Multiplicity Formula (see for instance [2, §8.2]) is also a quick corollary.

5. Proof of the Weyl Character Formula

The following proof is adapted from Igusa's notes [4]. For calculations it will be convenient to extend $\mathbf{Q}[\Lambda]$ to a larger ring $\mathbf{Q}[\frac{1}{2}\Lambda]$ by adjoining a square root $e(\frac{1}{2}\alpha)$ for each $\alpha \in \Phi$. We then complete $\mathbf{Q}[\frac{1}{2}\Lambda]$ to the algebra $\mathbf{Q}[[\frac{1}{2}\Lambda]]$ of formal power series generated by the $\mathrm{e}(\frac{1}{2}\lambda)$ for $\lambda \in \Lambda$. For example, in $\mathbf{Q}[[\frac{1}{2}\Lambda]]$ we have $\sum_{s=0}^{\infty} \mathrm{e}(\lambda)^s = \frac{1}{1-\mathrm{e}(\lambda)}$.

We shall also need the *Laplacian operator* $\Delta : \mathbf{Q}[[\frac{1}{2}\Lambda]] \to \mathbf{Q}[[\frac{1}{2}\Lambda]]$, defined by $\Delta\big(\mathrm{e}(\lambda)\big) = ||\lambda||^2 e(\lambda)$ for $\lambda \in \frac{1}{2}\Lambda$, and the bilinear form $\{\ ,\ \}$ on $\mathbf{Q}[[\frac{1}{2}\lambda]]$ defined by

$$\{\mathrm{e}(\lambda), e(\mu)\} = (\lambda, \mu)\mathrm{e}(\lambda + \mu) \quad \text{for } \lambda, \mu \in \tfrac{1}{2}\Lambda.$$

See Exercise 4.3(i) and (iv) for some motivation for Δ. These gadgets are related by the following lemma.

Lemma 5.1. *Let $f, g \in \mathbf{Q}[[\frac{1}{2}\Lambda]]$. Then*

$$\Delta(fg) = f\Delta(g) + \Delta(f)g + 2\{f, g\}.$$

Proof. By linearity it is sufficient to prove the lemma when $f = \mathrm{e}(\lambda)$ and $g = \mathrm{e}(\mu)$ for some $\lambda, \mu \in \frac{1}{2}\Lambda$. In this case it states that

$$||\lambda + \mu||^2 \mathrm{e}(\lambda + \mu) = \mathrm{e}(\lambda)||\mu||^2 \mathrm{e}(\mu) + ||\lambda^2||\mathrm{e}(\lambda)\mathrm{e}(\mu) + 2(\lambda, \mu)\mathrm{e}(\lambda + \mu)$$

which is obvious. $\square$

Proof of Weyl Character Formula. Let Q denote the denominator in the Weyl Character Formula. We begin the proof with Freudenthal's formula in the form

$$\big(||\mu+\delta||^2 - ||\delta||^2\big)n_\lambda = \big(||\lambda||^2 + (\lambda, 2\delta)\big)n_\lambda + 2\sum_{\alpha\in\Phi^+}\sum_{m=1}^{\infty}(\lambda+m\alpha,\alpha)n_{\lambda+m\alpha}.$$

Multiply both sides by $\mathrm{e}(\lambda)$ and sum over all $\lambda \in \Lambda$ to get

$$\big(||\mu+\delta||^2 - ||\delta||^2\big)\chi_V = \Delta(\chi_V) + \sum_{\lambda\in\Lambda}(\lambda, 2\delta)n_\lambda \mathrm{e}(\lambda) + X \tag{5}$$

where

$$\begin{aligned} X &= 2\sum_{\lambda\in\Lambda}\sum_{\alpha\in\Phi^+}\sum_{m=1}^{\infty}(\lambda+m\alpha,\alpha)n_{\lambda+m\alpha}\mathrm{e}(\lambda) \\ &= 2\sum_{\lambda\in\Lambda}\sum_{\alpha\in\Phi^+}\sum_{m=1}^{\infty}(\lambda,\alpha)n_{\lambda}\mathrm{e}(\lambda-m\alpha) \\ &= 2\sum_{\lambda\in\Lambda}\sum_{\alpha\in\Phi^+}(\lambda,\alpha)\frac{n_\lambda \mathrm{e}(\lambda)}{\mathrm{e}(\alpha)-1}. \end{aligned}$$

Now multiply through by Q and replace 2δ with $\sum_{\alpha\in\Phi^+}\alpha$ to combine the second two summands on the right-hand side of (5). This gives

$$\big(||\mu+\delta||^2 - ||\delta||^2\big)Q\chi_V = Q\Delta(\chi_V) + Q\sum_{\lambda\in\Lambda}\sum_{\alpha\in\Phi^+}(\lambda,\alpha)n_\lambda \mathrm{e}(\lambda)\frac{\mathrm{e}(\alpha)+1}{\mathrm{e}(\alpha)-1}.$$

Since $Q\chi_V$ is antisymmetric, it follows from Exercise 4.3(i) that $Q\chi_V = \sum_{w\in W}\varepsilon(w)w\cdot e(\mu+\delta)$ if and only if $\Delta(Q\chi_V) = ||\mu+\delta||^2 Q\chi_V$. Again by this exercise, $\Delta(Q) = ||\delta||^2 Q$. Hence it is sufficient to prove

$$\Delta(Q\chi_V) - \Delta(Q)\chi_V - Q\Delta(\chi_V) = Q\sum_{\lambda\in\Lambda}\sum_{\alpha\in\Phi^+}(\lambda,\alpha)n_\lambda \mathrm{e}(\lambda)\frac{\mathrm{e}(\alpha)+1}{\mathrm{e}(\alpha)-1}. \tag{6}$$

By Lemma 5.1, the left-hand side in (6) is $2\{Q,\chi_V\}$. So finally, it is sufficient to prove that

$$2\Big\{Q, \sum_{\lambda\in\Lambda} n_\lambda \mathrm{e}(\lambda)\Big\} = Q\sum_{\alpha\in\Phi^+}\frac{\mathrm{e}(\alpha)+1}{\mathrm{e}(\alpha)-1}\sum_{\lambda\in\Lambda}(\lambda,\alpha)n_\lambda \mathrm{e}(\lambda)$$

which, by linearity, follows from the relation

$$2\big\{Q, \mathrm{e}(\nu)\big\} = Q\sum_{\alpha\in\Phi^+}\frac{\mathrm{e}(\alpha)+1}{\mathrm{e}(\alpha)-1}(\nu,\alpha)e(\nu) \quad \text{for } \nu\in\Lambda,$$

proved in Exercise 5.2 below. □

Exercises

Exercise 1.1. Let $E = \langle u, v \rangle$ be the natural 2-dimensional $\mathsf{sl}_2(\mathbf{C})$-module. Show that $\mathrm{Sym}^d E$ is irreducible for each $d \in \mathbf{N}$.

Exercise 1.2. Let V be a finite-dimensional $\mathsf{sl}_2(\mathbf{C})$-module.

(i) Show that V contains an h-eigenvector v such that $e \cdot v = 0$.

(ii) Show that the submodule of V generated by V is d-dimensional if and only if $h \cdot v = dv$.

(iii) Deduce that any irreducible $\mathsf{sl}_2(\mathbf{C})$-module is isomorphic to $\mathrm{Sym}^d E$ for some $d \in \mathbf{N}_0$.

Exercise 2.1. Show that a Cartan subalgebra (as defined in Section 2) is abelian.

Solution. Given h, $k \in H$, we can write k as a sum of $\operatorname{ad} h$ eigenvectors, say $k = k_0 + \sum_{i=1}^n k_i$ where $(\operatorname{ad} h)k_0 = 0$ and $(\operatorname{ad} h)k_i = \lambda_i k_i$. Hence $(\operatorname{ad} h)^r k = \sum_{i=1}^n \lambda_i^r k_i$. A useful linear algebra lemma shows that all the k_i are in the Lie subalgebra of H generated by h and k. Now $[h, k_i] = \lambda_i k_i$ and so $(\operatorname{ad} k_i)^2 x = [k_i, [k_i, x]] = [k_i, -\lambda_i k_i] = 0$; since $k_i \in H$, $\operatorname{ad} k_i$ is diagonalizable, and so we must have $(\operatorname{ad} k_i)x = 0$. Hence $[h, k] = 0$.

Exercise 2.2. The aim of this exercise is to show that if H is a Cartan subalgebra of L then H is self-centralizing.

(i) Show that L_0 is nilpotent. [*Hint:* use Engel's theorem and the abstract Jordan decomposition.]

(ii) Show that there is a basis of L_0 in which all $\operatorname{ad} x : L \to L$ for $x \in L_0$ are represented by upper-triangular matrices.

(iii) Show that if $x \in L_0$ and $\operatorname{ad} x : L \to L$ is nilpotent then $\mathrm{Tr}(\operatorname{ad} x \circ \operatorname{ad} y) = 0$ for all $y \in L_0$. Deduce that $x = 0$.

(iv) Deduce that every element of L_0 is semisimple and hence show that $L_0 = H$.

Exercise 2.3. Let V be a complex vector space. Show that if x and $y \in \mathsf{gl}(V)$ are such that $[x, y]$ commutes with x then $[x, y]$ is nilpotent. [*Hint:* there is a quick solution using Lie's Theorem. For an *ad-hoc* proof (which then allows this exercise to be used as part of a proof of Lie's Theorem) first show that $\mathrm{Tr}[x, y]^n = 0$ for all $n \in \mathbf{N}$.]

Exercise 2.4. Let α and β be non-perpendicular roots in a root system. Use the fundamental relation (4) to find the possible angles between α and β and the possible values of $||\alpha||/||\beta||$.

Exercise 2.5. Find the Killing form of $\mathsf{sl}_2(\mathbf{C})$ with respect to the basis e, f, h and hence calculate $||\alpha||^2$ where α is the unique root of $\mathsf{sl}_2(\mathbf{C})$. (In practice the previous exercise always gives enough information, so this calculation is unnecessary. For example, this remark applies to Freudenthal's formula, since n_λ is expressed as a quotient of norms, and to Exercise 4.6, for the same reason.)

Exercise 2.6. Let V be a finite-dimensional L-module and let $v \in V$ be a highest-weight vector. Show that the submodule of L generated by v is irreducible.

Exercise 2.7. Let H be the Cartan subalgebra of diagonal matrices in $\mathsf{sl}_3(\mathbf{C})$. For $i \in \{1,2,3\}$, let $\varepsilon_i : H \to \mathbf{C}$ be the function sending $\mathrm{diag}(a_1, a_2, a_3)$ to a_i. Let $\alpha = \varepsilon_1 - \varepsilon_2$ and let $\beta = \varepsilon_2 - \varepsilon_3$.

- (i) Show that $\{\alpha, \beta\}$ is a base for the root system Φ.
- (ii) Show that $||\alpha|| = ||\beta||$ and that the angle between α and β is $2\pi/3$.
- (iii) Find the fundamental dominant integral weights ω_1, ω_2 corresponding to this base in terms of α and β.
- (iv) Show that $\omega_1 = \varepsilon_1$ and $\omega_2 = \varepsilon_1 + \varepsilon_2$. (Since $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ other, equivalent, expressions for ω_1 and ω_2 are also possible.)
- (iv) Express the highest weight of the natural, dual natural and adjoint representations of $\mathsf{sl}_3(\mathbf{C})$ as $\mathbf{Z}$-linear combinations of ω_1 and ω_2.

Exercise 3.1. Recall that $\delta = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$ and that $B = \{\alpha_1, \ldots, \alpha_\ell\}$ is a base for Φ.

- (i) Show that if $\beta \in \Phi^+$ and $\beta \neq \alpha_i$ then $S_{\alpha_i}(\beta) \in \Phi^+$
- (ii) Show that $S_{\alpha_i}(\delta) = \delta - \alpha_i$ for all i.
- (iii) Show that $\delta = \omega_1 + \cdots + \omega_\ell$ and deduce that $\delta \in \Lambda$.

Solution. (i) Since $\beta \neq \alpha_i$ and $k\alpha_i$ is a root if and only if $k \in \{+1, -1\}$ (see, for example, [1, Proposition 10.9]), there exists j such that α_j appears with a strictly positive coefficient in the expression for β as a $\mathbf{Z}$-linear combination of $\alpha_1, \ldots, \alpha_n$. Now α_j has the same coefficient in

$$S_{\alpha_i}(\beta) = \beta - \langle \beta, \alpha_i \rangle \alpha_i,$$

and so it follows that $S_{\alpha_i}(\beta) \in \Phi^+$.

(ii) Since S_{α_i} permutes $\Phi^+ \backslash \{\alpha_i\}$ and $S_{\alpha_i}(\alpha_i) = -\alpha_i$, we have

$$S_{\alpha_i}(\delta) = \tfrac{1}{2}\sum_{\beta \in \Phi} S_{\alpha_i}(\beta) = \tfrac{1}{2}\sum_{\beta \in \Phi} S_{\alpha_i}(\beta) - \alpha_i = \delta - \alpha_i$$

as required.

(iii) By definition $\langle \alpha_i, \omega_j \rangle = 0$ if $i \neq j$ and $\langle \alpha_i, \omega_j \rangle = 1$. Hence

$$S_{\alpha_j}(\sum_{i=1}^{\ell} \omega_i) = \sum_{i=1}^{\ell} \omega_i - \omega_j + S_{\alpha_j}(\omega_j) = \sum_{i=1}^{\ell} \omega_i - \omega_j + \omega_j - \alpha_j = \sum_{i=1}^{\ell} \omega_i - \alpha_j.$$

Hence by (ii), $-\delta + \sum_{i=1}^{\ell} \omega_i$ is invariant under the generators $S_{\alpha_1}, \ldots, S_{\alpha_\ell}$ of W. Hence $\delta = \sum_{i=1}^{\ell} \omega_i \in \Lambda$.

Exercise 3.2. Let $B : V \to V$ be a non-degenerate symmetric bilinear form on an n-dimensional vector space V. Suppose that $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are dual bases for V, so

$$B(x_i, y_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $\theta \in V^\star$ and let t_θ be the unique element such that $B(t_\theta, v) = \theta(v)$ for all $v \in V$. Let $v \in V$.

(i) Show that $v = \sum_{i=1}^{n} B(x_i, v) y_i = \sum_{j=1}^{n} B(v, y_j) x_j$.
(ii) Hence show that $B(t_\theta, t_\theta) = \sum_{k=1}^{n} \theta(x_k)\theta(y_k)$.

Solution. (i) For each j we have $B\big(\sum_{i=1}^{n} B(x_i, v) y_i, x_j\big) = B(x_j, v)$, hence $B\big(-v + \sum_{i=1}^{n} B(x_i, v) y_i, x_j\big) = 0$ for all j. Since $x_1, \ldots, x_n$ is a basis of V and B is non-degenerate, it follows that $v = \sum_{i=1}^{n} B(x_i, v) y_i, x_j$, as required. Similarly one finds that $v = \sum_{j=1}^{n} B(v, y_j) x_j$.

(ii) We have $t_\theta = \sum_{i=1}^{n} B(x_i, t_\theta) y_i$ and $t_\theta = \sum_{j=1}^{n} B(t_\theta, y_j) x_j$. Hence

$$(t_\theta, t_\theta) = \sum_{k=1}^{n} B(x_k, t_\theta) B(t_\theta, y_k) = \sum_{k=1}^{n} t_\theta(x_k) t_\theta(y_k)$$

as required.

Exercise 3.3. Prove Lemma 3.1. [*Hint:* Show that $\sum_{k=1}^{n} [x_k y_k, w] = \sum_{k=1}^{n} x_k [y_k w] + \sum_{k=1}^{n} [x_k w] y_k$ for $w \in L$, and then use Exercise 3.2(i) to express $[y_k, w]$ as a linear combination of $y_1, \ldots, y_n$ and $[x_k, w]$ as a linear combination of $x_1, \ldots, x_n$.]

Solution. Since $\mathcal{U}(L)$ is generated, as an algebra, by L, it is sufficient to prove that $[\sum_{k=1}^{n} x_k y_k, w] = 0$ for each $w \in L$. A routine calculation gives the result stated in the hint that

$$\sum_{k=1}^{n} [x_k y_k, w] = \sum_{k=1}^{n} x_k [y_k, w] + \sum_{k=1}^{n} [x_k, w] y_k.$$

By Exercise 3.2(i) we have $[y_k, w] = \sum_{i=1}^{n} \kappa(x_i, [y_k, w]) y_i$ and $[x_k, w] = \sum_{j=1}^{n} \kappa([x_k, w], y_j) x_j$. Substituting we get

$$\sum_{k=1}^{n} [x_k y_k, w] = \sum_{k=1}^{n} \sum_{i=1}^{n} \kappa(x_i, [y_k, w]) x_k y_i + \sum_{k=1}^{n} \sum_{j=1}^{n} \kappa([x_k, w], y_j) x_j y_k.$$

Now change the summation variables in the second sum and use the associativity of the Killing form to get

$$\begin{aligned}\sum_{k=1}^{n}[x_k y_k, w] &= \sum_{k=1}^{n}\sum_{i=1}^{n}\kappa(x_i,[y_k,w])x_k y_i + \sum_{i=1}^{n}\sum_{k=1}^{n}\kappa([x_i,w],y_k)x_k y_i \\ &= \sum_{k=1}^{n}\sum_{i=1}^{n}\bigl(-\kappa(x_i,[w,y_k]) + \kappa([x_i,w],y_k)\bigr)x_k y_i \\ &= 0\end{aligned}$$

as required.

Exercise 3.4. Take the notation from Lemma 3.3. Suppose that $\lambda(h_\alpha) \leq 0$.

(i) Deduce from (b) in Section 1 that if $U^{(i)}$ is a summand with *lowest* weight $(\lambda - b\alpha)(h_\alpha)$ where $b \in \mathbf{N}_0$, then $f_\alpha e_\alpha$ acts on $U^{(i)}_\lambda$ as the scalar $(b - \lambda(h_\alpha))(b+1)$.

(ii) Show that the number of summands $U^{(i)}$ with lowest weight $(\lambda - b\alpha)(h_\alpha)$ is $n_{\lambda - b\alpha} - n_{\lambda-(b+1)\alpha}$.

(iii) Hence show that $f_\alpha e_\alpha$ acts on V_λ as the scalar $-\sum_{b=0}^{\infty} n_{\lambda-b\alpha}\langle \lambda - b\alpha, \alpha\rangle$, as claimed in the proof of Lemma 3.3.

Solution. (i) If $U^{(i)}$ has lowest weight $(\lambda - b\alpha)(h_\alpha)$ then $U^{(i)}$ has highest weight $-(\lambda - b\alpha)(h_\alpha)$. If $v \in U^{(i)}_\lambda$ then

$$h \cdot v = \lambda(h) = (-\lambda - b\alpha)(h_\alpha) - 2(b - \lambda(h_\alpha))$$

and so taking $c = b - \lambda(h_\alpha)$ in (b) in Section 1 gives

$$\begin{aligned} f \cdot e \cdot v &= (b - \lambda(h_\alpha))\bigl((-\lambda - b\alpha)(h_\alpha) - (b - \lambda(h_\alpha)) + 1\bigr)v \\ &= (b - \lambda(h_\alpha))(b+1)v\end{aligned}$$

as required. Now (ii) follows from (a) in Section 1, in the same way as (c) did, and (iii) is an immediate corollary of (i) and (ii).

Exercise 3.5. Let ω_1, ω_2 be the fundamental dominant weights for $\mathsf{sl}_3(\mathbf{C})$ (see Exercise 2.7). Use Freudenthal's Formula to determine the dimensions of the weight spaces for the $\mathsf{sl}_3(\mathbf{C})$-module with highest weight $2\omega_1 + \omega_2$.

Exercise 4.1. Let $\tau : L \to \mathsf{gl}(V)$ be a representation of L. Let G be the simply connected Lie group corresponding to L and let $\rho : G \to \mathrm{GL}(V)$ be the corresponding representation of G, as defined by

$$\rho(\exp x) = \exp\bigl(\tau(x)\bigr) \quad \text{for } x \in L.$$

(This defines ρ on a generating set for G.) Let $\lambda \in \Lambda$. Show that if $h \in H$ and $v \in V_\lambda$ then $\rho(\exp h)v = \exp\bigl(\lambda(h)\bigr)v$.

Exercise 4.2. Show that if V is an L-module then $\chi_V \in \mathbf{Q}[\Lambda]$ is symmetric.

Exercise 4.3. Let Λ_{dom} be the set of *strictly* dominant weights in Λ.

(i) Given $\lambda \in \Lambda$ define $a(\lambda) = \sum_{w\in W} \varepsilon(w)\, w{\cdot}\mathrm{e}(\lambda)$. Show that $\Delta\big(a(\lambda)\big) = ||\lambda||^2 a(\lambda)$ and deduce that $\{a(\lambda) : \lambda \in \Lambda_{\text{dom}}\}$ is a $\mathbf{Z}$-basis of Δ-eigenvectors for the set of all antisymmetric elements of $\mathbf{Q}[\Lambda]$.

(ii) Show that

$$\mathrm{e}(-\delta) \prod_{\alpha\in\Phi^+} \big(\mathrm{e}(\alpha) - 1\big) = \prod_{\alpha\in\Phi^+} \big(\mathrm{e}(\tfrac{1}{2}\alpha) - \mathrm{e}(-\tfrac{1}{2}\alpha)\big)$$

and that either side is antisymmetric.

(iii) Show that

$$\sum_{w\in W} \varepsilon(w)\, w \cdot \mathrm{e}(\delta) = \prod_{\alpha\in\Phi^+} \big(\mathrm{e}(\tfrac{1}{2}\alpha) - \mathrm{e}(-\tfrac{1}{2}\alpha)\big)$$

(iv) Prove that $f \in \mathbf{Q}[\frac{1}{2}\Lambda]$ is antisymmetric if and only if

$$f = g \prod_{\alpha\in\Phi^+} \big(\mathrm{e}(\tfrac{1}{2}\alpha) - \mathrm{e}(-\tfrac{1}{2}\alpha)\big)$$

for some symmetric g.

Solution. (i) Fix a total order on Λ refining the dominance order. Define the *degree* of an antisymmetric element f to be the greatest weight μ in this order such that $e(\mu)$ has a non-zero coefficient in f. If μ is the greatest weight of f then $\mu \in \Lambda_{\text{dom}}$ and μ is acted on regularly by the Weyl group. Hence $f - \sum_{w\in W} \varepsilon(w)\, w \cdot e(\mu)$ has strictly smaller weight. The result now follows by induction.

(ii) The equality is routine. Recall that $\{\alpha_1, \ldots, \alpha_\ell\}$ is a base for Φ. It follows from Exercise 3.1(i) and (ii) that

$$\begin{aligned} S_{\alpha_i}\Big(\prod_{\alpha\in\Phi^+} \big(\mathrm{e}(\tfrac{1}{2}\alpha) - \mathrm{e}(-\tfrac{1}{2}\alpha)\big)\Big) &= \frac{-\mathrm{e}(\frac{1}{2}\alpha_i) + \mathrm{e}(-\frac{1}{2}\alpha_i)}{\mathrm{e}(\frac{1}{2}\alpha_i) - \mathrm{e}(-\frac{1}{2}\alpha_i)} \prod_{\alpha\in\Phi^+} \big(\mathrm{e}(\tfrac{1}{2}\alpha) - \mathrm{e}(-\tfrac{1}{2}\alpha)\big) \\ &= - \prod_{\alpha\in\Phi^+} \big(\mathrm{e}(\tfrac{1}{2}\alpha) - \mathrm{e}(-\tfrac{1}{2}\alpha)\big). \end{aligned}$$

Hence the right-hand side is antisymmetric.

(iii) Both sides are anti-symmetric and the coefficients of $\mathrm{e}(\delta)$ agree. The result now follows from (i) since, by Exercise 3.1(iii), δ is the smallest element of Λ_{dom}.

(iv) *Sketch:* it is sufficient to prove that each $a(\lambda)$ is divisible by $\prod_{\alpha\in\Phi^+} \big(\mathrm{e}(\frac{1}{2}\alpha) - \mathrm{e}(-\frac{1}{2}\alpha)\big)$. This follows using that $\mathbf{Q}[\frac{1}{2}\Lambda]$ is a UFD.

Exercise 4.4. Let ω be the unique fundamental dominant weight for $\mathsf{sl}_2(\mathbf{C})$, so $\omega \in \langle h\rangle^\star$ is defined by $\omega(h) = 1$.

(i) Use the results of Section 1 to show that V is the irreducible $\mathsf{sl}_2(\mathbf{C})$-module with highest weight $d\omega$ then

$$\chi_V = e(d\omega) + \mathrm{e}((d-2)\omega) + \cdots + \mathrm{e}(-d\omega).$$

(ii) Check that this is consistent with the Weyl Character Formula.

Exercise 4.5. Let ω_1, ω_2 be the fundamental dominant weights for $\mathsf{sl}_3(\mathbf{C})$ (see Exercise 2.7).

(i) Use the Weyl Character Formula to determine the characters of the finite-dimensional irreducible $\mathsf{sl}_3(\mathbf{C})$-module V with highest weight $a\omega_1 + b\omega_2$ where $a, b \in \mathbf{N}_0$.

(ii) Give a necessary and sufficient condition on a and b for V to have a weight space of dimension at least two.

Exercise 4.6. Deduce from the Weyl Character Formula that if V is the irreducible L-module with highest weight λ then

$$\dim V = \frac{\prod_{\alpha\in\Phi^+}(\lambda+\delta,\alpha)}{\prod_{\alpha\in\Phi^+}(\lambda,\alpha)}$$

Exercise 5.1. Show that if $f, g, h \in \mathbf{Q}[[\Lambda]]$ then $\{fg, h\} = f\{g,h\} + \{f,h\}g$.

Exercise 5.2. Recall that Q is the denominator in the Weyl Character Formula. Use Exercise 4.3(iii) and Exercise 5.1 to show that

$$2\{Q, \mathrm{e}(\nu)\} = Q \sum_{\alpha\in\Phi^+} \frac{\mathrm{e}(\alpha)+1}{\mathrm{e}(\alpha)-1}(\nu,\alpha)\mathrm{e}(\nu)$$

Solution. By the generalization of Exercise 5.1 to arbitrary products we have

$$\begin{aligned}
2\{Q, \mathrm{e}(\nu)\} &= 2\{\prod_{\alpha\in\Phi^+} \frac{1}{\mathrm{e}(\frac{1}{2}\alpha)-\mathrm{e}(-\frac{1}{2}\alpha)}, \mathrm{e}(\nu)\} \\
&= 2\sum_{\alpha\in\Phi^+} \frac{Q}{\mathrm{e}(\frac{1}{2}\alpha)-\mathrm{e}(-\frac{1}{2}\alpha)} \{\mathrm{e}(\tfrac{1}{2}\alpha)-\mathrm{e}(-\tfrac{1}{2}\alpha), \mathrm{e}(\nu)\} \\
&= 2\sum_{\alpha\in\Phi^+} \frac{Q}{\mathrm{e}(\frac{1}{2}\alpha)-\mathrm{e}(-\frac{1}{2}\alpha)} \Big((\tfrac{1}{2}\alpha,\nu)\mathrm{e}(\nu+\tfrac{1}{2}\alpha) + (\tfrac{1}{2}\alpha,\nu)\mathrm{e}(\nu-\tfrac{1}{2}\alpha)\Big) \\
&= \sum_{\alpha\in\Phi^+} \frac{Q}{\mathrm{e}(\frac{1}{2}\alpha)-\mathrm{e}(-\frac{1}{2}\alpha)} (\alpha,\nu)\big(\mathrm{e}(\tfrac{1}{2}\alpha)+\mathrm{e}(-\tfrac{1}{2}\alpha)\big)\mathrm{e}(\nu) \\
&= Q\sum_{\alpha\in\Phi^+} \frac{\mathrm{e}(\frac{1}{2}\alpha)+\mathrm{e}(-\frac{1}{2}\alpha)}{\mathrm{e}(\frac{1}{2}\alpha)-\mathrm{e}(-\frac{1}{2}\alpha)}(\nu,\alpha)\mathrm{e}(\nu) \\
&= Q\sum_{\alpha\in\Phi^+} \frac{\mathrm{e}(\alpha)+1}{\mathrm{e}(\alpha)-1}(\nu,\alpha)\mathrm{e}(\nu)
\end{aligned}$$

as required.

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