NOTES ON THE WEYL CHARACTER FORMULA

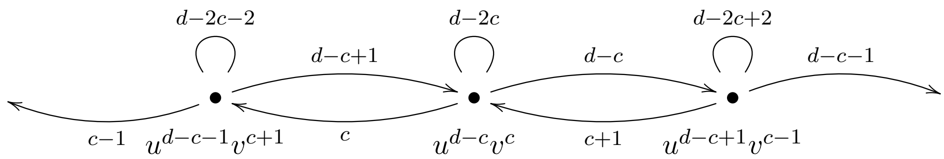
The aim of these notes is to give a self-contained algebraic proof of the Weyl Character Formula. The necessary background results on modules for $\mathsf{sl}_2(\mathbf{C})$ and complex semisimple Lie algebras are outlined in the first two sections. Some technical details are left to the exercises at the end; solutions are provided when the exercise is needed for the proof.

1. Representations of $\mathsf{sl}_2(\mathbf{C})$

Define

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and note that $\langle h, e, f \rangle = \mathsf{sl}_2(\mathbf{C})$. Let u, v be the canonical basis of $E = \mathbf{C}^2$. Then each $\mathrm{Sym}^d E$ is irreducible with u^d spanning the highest-weight space of weight d and, up to isomorphism, $\mathrm{Sym}^d E$ is the unique irreducible $\mathsf{sl}_2(\mathbf{C})$-module with highest weight d. (See Exercises 1.1 and 1.2.) The diagram below shows the actions of h, e and f on the canonical basis of $\mathrm{Sym}^d E$: loops show the action of h, arrows to the right show the action of e, arrows to the left show the action of f.



In particular

(a) the eigenvalues of h on $\mathrm{Sym}^d E$ are $-d, -d+2, \ldots, d-2, d$ and each h-eigenspace is 1-dimensional,

(b) if $w \in \mathrm{Sym}^d E$ and $h \cdot w = (d-2c)w$ then $f \cdot e \cdot w = c(d-c+1)w$.

If V is an arbitrary $\mathsf{sl}_2(\mathbf{C})$-module then, by Weyl's Theorem (see [2, Appendix B] or [4, §6.3]), V decomposes as a direct sum of irreducible $\mathsf{sl}_2(\mathbf{C})$-submodules. Let $V_r = \{v \in V : h \cdot w = rv\}$ for $r \in \mathbf{Z}$. Then (a) implies

(c) if $r \geq 0$ then the number of irreducible summands of V with highest weight r is $\dim V_r - \dim V_{r+2}$.

2. Prerequisites on complex semisimple Lie algebras

In this section we recall the basic setup of a Cartan subalgebra H inside a complex semisimple Lie algebra L, a lattice of weights $\Lambda \subseteq H^\star_{\mathbf{R}}$ and a root system $\Phi \subseteq \Lambda$. The mathematically most interesting parts are that H is self-centralizing (see Exercise 2.2) and the trick used to construct an

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$\mathsf{sl}_2(\mathbf{C})$-subalgebra corresponding to each root. For an example of all the theory below, see Exercise 2.8 and the applications to sl_n and sp_{2n} in §6.

Cartan subalgebras. We define a *Cartan subalgebra* of L to be a Lie subalgebra H of L maximal subject to the condition that $\operatorname{ad} h : L \to L$ is diagonalizable for all $h \in H$. It is an interesting fact (see Exercise 2.1) that any Cartan subalgebra is abelian. We may therefore decompose L as a direct sum of simultaneous eigenspaces for the elements of H. To each simultaneous eigenspace V we associate the unique $\alpha \in H^\star$ such that $(\operatorname{ad} h)x = \alpha(h)x$ for all $h \in H$ and $x \in V$. For $\alpha \in H^\star$ let

$$L_\alpha = \{x \in L : (\operatorname{ad} h)x = \alpha(h)x \text{ for all } h \in H,\, x \in V\}$$

and let Φ be the set of all non-zero $\alpha \in H^\star$ such that $L_\alpha \neq 0$. The elements of Φ are called *roots* and L_α is the *root space* corresponding to $\alpha \in \Phi$ and we have

$$L = L_0 \oplus \Big(\bigoplus_{\alpha \in \Phi} L_\alpha \Big).$$

Note that L_0 is the centralizer of H in L. It is an important and non-obvious fact (see Exercise 2.2) that $L_0 = H$, so H is self-centralizing: An easy calculation shows that

$$[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta} \quad \text{for all } \alpha, \beta \in \Phi_0. \tag{2.1}$$

Killing form. The *Killing form* on L is the bilinear form $\kappa(x,y) = \operatorname{Tr}(\operatorname{ad} x \circ \operatorname{ad} y)$. By Cartan's Criterion κ is non-degenerate. It follows from (2.1) that if $x \in L_\alpha$ and $y \in L_\beta$ where $\alpha, \beta \in \Phi_0$, then $\operatorname{ad} x \circ \operatorname{ad} y$ is nilpotent, unless $\alpha + \beta = 0$. Therefore if $\alpha, \beta \in \Phi_0$ then $L_\alpha \perp L_\beta$ unless $\beta = -\alpha$. Hence α is a root if and only if $-\alpha$ is a root and the restriction of κ to $L_\alpha \times L_{-\alpha}$ is non-degenerate. In particular, the restriction of κ to $H \times H$ is non-degenerate. For each $\alpha \in \Phi$, let $t_\alpha \in H$ be the unique element of H such that

$$\kappa(t_\alpha, h) = \alpha(h) \quad \text{for all } h \in H.$$

sl_2 *subalgebras.* Choose $e \in L_\alpha$ and $f \in L_{-\alpha}$ such that $\kappa(e,f) \neq 0$. By the associativity of the Killing form

$$\kappa(h, [e,f]) = \kappa([h,e], f) = \alpha(h)\kappa(e,f) \quad \text{for all } h \in H.$$

Since κ is non-degenerate on H, there exists $h \in H$ such that $\alpha(h) = \kappa(t_\alpha, h) \neq 0$. Since $\kappa(e,f) \neq 0$, the previous equation then implies that $[e,f] \neq 0$. Consider the Lie subalgebra

$$\langle e, f, [e,f] \rangle$$

of L. Since $[e,f] \in [L_\alpha, L_{-\alpha}] \subseteq H$ we have $\big[[e,f], e\big] = \alpha([e,f])e$ and $\big[[e,f], f\big] = -\alpha([e,f])f$.

If $\alpha([e,f]) = 0$ then $[e,f]$ is central in $\langle e, f, [e,f] \rangle$. By Exercise 2.3 below $[e,f]$ is nilpotent. But $[e,f] \in H$ and all the elements of H are semisimple. So $[e,f] = 0$, which contradicts the previous paragraph. Therefore

$\alpha([e,f]) \neq 0$ and we can scale e so that $\alpha([e,f]) = 2$ and so $\langle e, f, [e,f]\rangle \cong \mathsf{sl}_2(\mathbf{C})$.

For each $\alpha \in \Phi$ let $\langle e_\alpha, f_\alpha, h_\alpha\rangle$ be a subalgebra of L constructed as above so that

$$[e_\alpha, f_\alpha] = h_\alpha, \quad [h_\alpha, e_\alpha] = 2e_\alpha, \quad [h_\alpha, f_\alpha] = 2f_\alpha. \tag{2.2}$$

We may suppose that these elements are chosen so that $e_{-\alpha} = f_\alpha$ and $f_{-\alpha} = e_\alpha$ for each $\alpha \in \Phi$.

Relationship between t_α and h_α. By choice of t_α we have $\kappa(t_\alpha, h) = \alpha(h)$ for all $h \in H$. By associativity of the Killing form we also have

$$\kappa([e_\alpha, f_\alpha], h) = \kappa(e_\alpha, [f_\alpha, h]) = \kappa(e_\alpha, \alpha(h) f_\alpha) = \alpha(h)\kappa(e_\alpha, f_\alpha).$$

Hence

$$\kappa\Big(t_\alpha - \frac{[e_\alpha, f_\alpha]}{\kappa(e_\alpha, f_\alpha)}, h\Big) = 0 \quad \text{for all } h \in H.$$

Since the restriction of κ to $H \times H$ is non-degenerate it follows that

$$t_\alpha = \frac{h_\alpha}{\kappa(e_\alpha, f_\alpha)}. \tag{2.3}$$

Since $\kappa(t_\alpha, t_\alpha) = \alpha(t_\alpha)$, this implies the useful relations

$$2 = \alpha(h_\alpha) = \kappa(t_\alpha, h_\alpha) = \frac{\kappa(h_\alpha, h_\alpha)}{\kappa(e_\alpha, f_\alpha)} = \kappa(e_\alpha, f_\alpha)\kappa(t_\alpha, t_\alpha). \tag{2.4}$$

Transport of the Killing form to $H^\star_{\mathbf{R}}$. Observe that if Φ does not span $H^\star$ then there exists $h \in H$ such that $\alpha(h) = 0$ for all $\alpha \in \Phi$, and so $[h, L_\alpha] = 0$ for all $\alpha \in \Phi$. By Exercise 2.2 we deduce that $h \in Z(L)$, which contradicts the assumption that L is semisimple. (See Exercise 2.4 for an alternative argument when L is simple.) Hence there is a unique bilinear form $(\ ,\ )$ on $H^\star$ such that

$$(\alpha, \beta) = \kappa(t_\alpha, t_\beta) \quad \text{for } \alpha, \beta \in \Phi.$$

By (2.3) and (2.4) we have the fundamental formula

$$\frac{2(\alpha,\beta)}{(\beta,\beta)} = \kappa\Big(t_\alpha, \frac{2t_\beta}{\kappa(t_\beta, t_\beta)}\Big) = \kappa(t_\alpha, h_\beta) = \alpha(h_\beta). \tag{2.5}$$

Note also that $\alpha(h_\beta)$ is an eigenvalue of h_β in the finite-dimensional $\mathsf{sl}(\beta)$-module L. It follows that $(\ ,\ )$ takes real values on the roots and from the equation $\kappa(h,k) = \sum_{\alpha\in\Phi} \alpha(h)\alpha(k)$ for $h, k \in H$, we see that it is a real-valued inner-product on $H^\star_{\mathbf{R}} = \langle \alpha : \alpha \in \Phi\rangle_{\mathbf{R}}$. Exercise 2.5 shows that the angles between the roots are determined by (2.5). (In fact if L is a simple Lie algebra then Φ is a connected root system and $(\ ,\ )$ is completely determined by (2.5) and (α,α) for any single root $\alpha \in \Phi$.)

Angled brackets notation. It will be convenient to define

$$\langle \lambda, \mu \rangle = \frac{2(\lambda, \mu)}{(\mu, \mu)}$$

for $\lambda, \mu \in H^\star_{\mathbf{R}}$. Note that the form $\langle\ ,\ \rangle$ is linear only in its first component. This notation will often be used when $\mu \in \Phi$, in which case (2.5) implies that $\langle \lambda, \beta \rangle = \lambda(h_\beta)$.

Fundamental dominant weights. Recall that $\{\alpha_1, \ldots, \alpha_\ell\}$ is a base for Φ if element of Φ can be written uniquely as either a non-negative or non-positive integral linear combination of the α_i. (For a proof that every root system has a basis, see [2, Theorem 11.10] or [4, Theorem 10.1].) Fix, once and for all, a base $\{\alpha_1, \ldots, \alpha_\ell\}$ for Φ and let Φ^+ be the set of positive roots with respect to this basis. There exist unique $\omega_1, \ldots, \omega_\ell \in H^\star$ such that, for all $i, j \in \{1, \ldots, \ell\}$,

$$\omega_i(h_{\alpha_j}) = [i = j] \tag{2.6}$$

where $[i = j]$ is the Iverson bracket, equal to 1 if $i = j$ and 0 otherwise.

$$\Lambda = \langle \omega_1, \ldots, \omega_\ell \rangle_{\mathbf{Z}} \subseteq H^\star. \tag{2.7}$$

Weight space decomposition. The elements of H act semisimply in any finite-dimensional L-module (see [4, Corollary 6.3]). By Section 1, the eigenvalues of each h_{α_j} are integral. Hence if V is a finite-dimensional L-module then

$$V\!\downarrow_H = \bigoplus_{\lambda \in \Lambda} V_\lambda$$

where

$$V_\lambda = \{v \in V : h \cdot v = \lambda(h)v \text{ for all } h \in H\}.$$

(The root spaces defined earlier are weight spaces for the action of L on itself by the adjoint representation.) We shall say that an element of V lying in some non-zero V_λ is a *weight vector.* Starting with any weight vector, and then repeatedly applying the elements e_α for $\alpha \in \Phi^+$, it follows that V contains a weight vector v such that $e_\alpha \cdot v = 0$ for all $\alpha \in \Phi^+$. Such a vector is said to be a *highest-weight vector* with respect to the base $\{\alpha_1, \ldots, \alpha_\ell\}$. By Exercise 2.7, the submodule of V generated by a highest weight vector is irreducible.

3. Freudenthal's Formula

Recall that the lattice of weights Λ was defined in (2.7). Let $V(\mu)$ be an irreducible L-module of highest weight $\mu \in \Lambda$. Set $n(\mu)_\nu = \dim V(\mu)_\nu$ for each $\nu \in \Lambda$. The aim of this section is to prove *Freudenthal's Formula*, that if $\lambda \in \Lambda$ then

$$\big(||\mu + \delta||^2 - ||\lambda + \delta||^2\big) \dim n(\mu)_\lambda = 2 \sum_{\alpha \in \Phi^+} \sum_{m=1}^{\infty} n(\mu)_{\lambda + m\alpha} (\lambda + m\alpha, \alpha) \tag{3.1}$$

where

$$\delta = \tfrac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \in \tfrac{1}{2}\Lambda \tag{3.2}$$

is half the sum of the positive roots. By Exercise 3.1 we have $\delta \in \Lambda$. See Exercises 6.1 and 6.3 for δ in the type A and C cases.

The key idea in this proof (which is based on [6, VIII.2]) is to calculate the scalar by which a central element in the universal enveloping algebra $\mathcal{U}(L)$ acts on V, using the theory of $\mathsf{sl}_2(\mathbf{C})$-modules in Section 1. The following lemma gives a construction of such central elements.

Lemma 3.1. *Suppose that $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are bases of L such that*

$$\kappa(x_i, y_j) = [i = j]$$

Then $\sum_{i=1}^n x_i y_i$ is in the centre of $\mathcal{U}(L)$.

Proof. See Exercise 3.3. □

Let $\alpha, \beta \in \Phi$. By (2.1) we have $\kappa(e_\alpha, f_\beta) = 0$ whenever $\alpha \neq \beta$ and by (2.4) we have $\kappa(e_\alpha, f_\alpha) = 2/\kappa(t_\alpha, t_\alpha) = 2/(\alpha, \alpha)$ and $\kappa(t_\alpha, h_\alpha) = 2$ for all $\alpha \in \Phi$. Lemma 3.1 therefore implies that

$$\Gamma = \sum_{\alpha \in \Phi} \frac{(\alpha, \alpha)}{2} f_\alpha e_\alpha + \frac{1}{2} \sum_{j=1}^{\ell} t_{\alpha_j} h_{\alpha_j}$$

is in the centre of $\mathcal{U}(L)$. As noted after (2.2) we have chosen root-space elements so that $e_{-\alpha} = f_\alpha$ and $f_{-\alpha} = e_\alpha$ for each $\alpha \in \Phi^+$. Hence $f_{-\alpha} e_{-\alpha} = e_\alpha f_\alpha = h_\alpha + f_\alpha e_\alpha$ and

$$\Gamma = \sum_{\alpha \in \Phi^+} \frac{(\alpha, \alpha)}{2} h_\alpha + \sum_{\alpha \in \Phi^+} (\alpha, \alpha) f_\alpha e_\alpha + \frac{1}{2} \sum_{j=1}^{\ell} t_{\alpha_j} h_{\alpha_j}.$$

The action of each of the three summands of Γ preserves the weight spaces $V(\mu)_\lambda$. The next three lemmas determine the traces of these summands on each $V(\mu)_\lambda$. The first explains the appearance of δ in Freudenthal's Formula.

Lemma 3.2. *If $\lambda \in \Lambda$ and $v \in V(\mu)_\lambda$ then*

$$\sum_{\alpha \in \Phi^+} \frac{(\alpha, \alpha)}{2} h_\alpha \cdot v = (\lambda, 2\delta) v.$$

Proof. Using (2.5) we get

$$\sum_{\alpha \in \Phi^+} \frac{(\alpha, \alpha)}{2} \lambda(h_\alpha) = \sum_{\alpha \in \Phi^+} \frac{(\alpha, \alpha)}{2} \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} = \sum_{\alpha \in \Phi^+} (\lambda, \alpha) = (\lambda, 2\delta)$$

as required. □

Lemma 3.3. *If $\alpha \in \Phi$ and $\lambda \in \Lambda$ then*

$$(\alpha, \alpha) \operatorname{Tr}_{V(\mu)_\lambda}(f_\alpha e_\alpha) = 2 \sum_{m=1}^{\infty} n(\mu)_{\lambda + m\alpha} (\lambda + m\alpha, \alpha).$$

Proof. Since $\dfrac{2(\lambda+m\alpha,\alpha)}{(\alpha,\alpha)} = \langle \lambda + m\alpha, \alpha\rangle$, it is equivalent to prove that

$$\mathrm{Tr}_{V(\mu)_\lambda}(f_\alpha e_\alpha) = \sum_{m=1}^{\infty} n(\mu)_{\lambda+m\alpha}\langle \lambda + m\alpha, \alpha\rangle.$$

Let $W = \bigoplus_{c\in\mathbf{Z}} V(\mu)_{\lambda+c\alpha}$. Note that W is a direct sum of weight spaces for the action of H, and that W is an $\mathsf{sl}(\alpha)$-submodule of V. We may write

$$W = U^{(1)} \oplus \cdots \oplus U^{(d)}$$

where each $U^{(i)}$ is an irreducible $\mathsf{sl}(\alpha)$-module.

Assume first of all that $\lambda(h_\alpha) \geq 0$. Suppose that $U^{(i)}_\lambda \neq 0$. Choose m maximal such that $U^{(i)}_{\lambda+m\alpha} \neq 0$. Then $U^{(i)}$ has highest weight $(\lambda+m\alpha)(h_\alpha)$ as an $\mathsf{sl}(\alpha)$-module and by (b) in Section 1, the scalar by which $f_\alpha e_\alpha$ acts on a vector in $U^{(i)}_\lambda$ is

$$m\big((\lambda+m\alpha)(h_\alpha) - m + 1\big) = m\big(\lambda(h_\alpha) + m + 1\big).$$

It follows from (c) in Section 1 that the number of summands $U^{(i)}$ with highest weight $(\lambda+m\alpha)(h_\alpha)$ as an $\mathsf{sl}(\alpha)$-module is $n(\mu)_{\lambda+m\alpha} - n(\mu)_{\lambda+(m+1)\alpha}$. Hence

$$\begin{aligned}
\mathrm{Tr}_{V(\mu)_\lambda}(f_\alpha e_\alpha) &= \sum_{m=0}^{\infty}\big(n(\mu)_{\lambda+m\alpha} - n(\mu)_{\lambda+(m+1)\alpha}\big) m(\lambda(h_\alpha)+m+1) \\
&= \sum_{m=1}^{\infty} n(\mu)_{\lambda+m\alpha}\big(m(\lambda(h_\alpha)+m+1) - (m-1)(\lambda(h_\alpha)+m)\big) \\
&= \sum_{m=1}^{\infty} n(\mu)_{\lambda+m\alpha}(\lambda(h_\alpha)+2m).
\end{aligned}$$

as required. Note that this equation holds even when $V(\mu)_\lambda = 0$, since the argument just given shows that both sides are zero.

If $\lambda(h_\alpha) \leq 0$ then a similar calculation (see Exercise 3.4) shows that $f_\alpha e_\alpha$ acts as the scalar $-\sum_{b=0}^{\infty} n(\mu)_{\lambda-b\alpha}\langle \lambda - b\alpha, \alpha\rangle$ on $V(\mu)_\lambda$. Now

$$\sum_{c=-\infty}^{\infty} n(\mu)_{\lambda+c\alpha}\langle \lambda + c\alpha, \alpha\rangle = 0$$

since each irreducible summand $U^{(i)}$ contributes the sum of the h_α eigenvalues on $U^{(i)}$, which is 0 by (a) in Section 1. Adding these two equations we get the required formula. □

Lemma 3.4. *Let $\lambda \in \Lambda$. If $v \in V(\mu)_\lambda$ then*

$$\frac{1}{2}\sum_{j=1}^{\ell} t_{\alpha_j} h_{\alpha_j} \cdot v = (\lambda,\lambda)v$$

Proof. We saw earlier that $\frac{1}{2}t_{\alpha_1}, \ldots, \frac{1}{2}t_{\alpha_\ell}$ and $h_{\alpha_1}, \ldots, h_{\alpha_\ell}$ are dual bases of $H^\star$ with respect to the Killing form κ on $H \times H$. By Exercise 3.2(ii)

$$\frac{1}{2}\sum_{i=1}^{\ell} \lambda(t_{\alpha_j})\lambda(h_{\alpha_j}) = (\lambda, \lambda)$$

as required. □

Since Γ is central in $\mathcal{U}(L)$ it acts as a scalar on V, say γ. Let $\lambda \in \Lambda$. By Lemmas 3.2, 3.3 and 3.4, we have

$$\begin{aligned} n(\mu)_\lambda \gamma &= \mathrm{Tr}_{V(\mu)_\lambda}(f_\alpha e_\alpha) \\ &= (\lambda, 2\delta)n(\mu)_\lambda + 2\sum_{\alpha \in \Phi^+}\sum_{m=1}^{\infty} n(\mu)_{\lambda+m\alpha}(\lambda + m\alpha, \alpha) + (\lambda, \lambda)n(\mu)_\lambda. \end{aligned}$$

Since $e_\alpha \cdot V(\mu)_\mu = 0$ for all $\alpha \in \Phi^+$, $n(\mu)_\mu = 1$, and $(\lambda, 2\delta) + (\lambda, \lambda) = ||\lambda + \delta||^2 - ||\delta||^2$, the previous equation implies

$$\gamma = ||\mu + \delta||^2 - ||\delta^2||.$$

Comparing these two equations we obtain

$$\big(||\mu + \delta||^2 - ||\lambda + \delta||^2\big)n(\mu)_\lambda = 2\sum_{\alpha \in \Phi^+}\sum_{m=1}^{\infty} n(\mu)_{\lambda+m\alpha}(\lambda + m\alpha, \alpha)$$

as stated in Freudenthal's Formula. For an immediate application of Freudenthal's Formula see Exercise 3.5 in the final exercise section.

4. Statement of Weyl Character Formula

Formal exponentials and characters. For each $\lambda \in \Lambda$ we introduce a formal symbol $e(\lambda)$ which we call the *formal exponential of* λ. Let $\mathbf{Q}[\Lambda]$ denote the abelian group with $\mathbf{Z}$-basis $\{e(\lambda) : \lambda \in \Lambda\}$. We make $\mathbf{Q}[\Lambda]$ into an algebra by defining the multiplication on basis elements by

$$\mathrm{e}(\lambda)e(\lambda') = \mathrm{e}(\lambda + \lambda') \quad \text{for } \lambda, \lambda' \in \Lambda.$$

Note that $\mathrm{e}(0) = 1 \in \mathbf{Q}$ and that each $\mathrm{e}(\lambda)$ is invertible, with inverse $\mathrm{e}(-\lambda)$. This definition is motivated by one-parameter subgroups: see Exercise 4.1. (We mention that $\mathbf{Q}[\Lambda]$ is the group algebra of the abelian group Λ, using multiplicative notation for group multiplication to distinguish it from the algebra addition.) Given an L-module V, we define the *formal character* of L by

$$\chi_V = \sum_{\lambda \in \Lambda} (\dim V_\lambda) e(\lambda) \in \mathbf{Q}[\Lambda].$$

Weyl group. Let $S_\beta : H^\star_{\mathbf{R}} \to H^\star_{\mathbf{R}}$ denote the reflection corresponding to $\beta \in \Phi$ as defined by

$$S_\beta(\theta) = \theta - \frac{2(\theta,\beta)}{(\beta,\beta)}\beta \quad \text{for } \theta \in H^\star_{\mathbf{R}}.$$

The alterative forms $S_\beta(\theta) = \theta - \langle \theta, \beta \rangle \beta = w - \theta(h_\beta)\alpha$ are often useful. By definition the *Weyl group* of L is the group generated by the S_β for $\beta \in \Phi$. We define $\operatorname{sgn}(w) = 1$ if w is a product of an even number of reflections, and $\operatorname{sgn}(w) = -1$ otherwise. The Weyl group W acts on $\mathbf{Q}[\Lambda]$ by $w \cdot \mathrm{e}(\lambda) = e(w(\lambda))$ for $w \in W$ and $\lambda \in \Lambda$. We shall see in §6 that in the type A case for $\mathsf{sl}_n(\mathbf{C})$ we may identify W with the symmetric group $\mathfrak{S}_n$; this justifies the notation $\operatorname{sgn}(w)$ used above. In this case the symmetric elements in the following definition can be identified with symmetric polynomials.

Symmetric and antisymmetric elements.

Definition 4.1. We say that an element $f \in \mathbf{Q}[\Lambda]$ is *symmetric* if $w \cdot f = f$ for all $w \in W$ and *antisymmetric* if $w \cdot f = \operatorname{sgn}(w) f$ for all $w \in W$.

By Exercise 4.3(iv), $f \in \mathbf{Q}[\Lambda]$ is antisymmetric if and only if

$$f = g \sum_{w \in W} \operatorname{sgn}(w)\, w \cdot \mathrm{e}(\delta)$$

for some symmetric g.

Weyl Character Formula. We may now state the main result. By the result on antisymmetric elements of $\mathbf{Q}[\Lambda]$ just mentioned, the right-hand side in the formula below is a well-defined symmetric element of $\mathbf{Q}[\Lambda]$.

Theorem 4.2 (Weyl Character Formula)**.** *Let $V(\mu)$ be the irreducible L-module of highest weight $\mu \in \Lambda$. Then*

$$\chi_{V(\mu)} = \frac{\sum_{w \in W} \operatorname{sgn}(w)\, w \cdot \mathrm{e}(\mu + \delta)}{\sum_{w \in W} \operatorname{sgn}(w)\, w \cdot \mathrm{e}(\delta)}.$$

We prove Weyl's Character Formula using Frendenthal's Formula (3.1) in §5 below, after proving a notable corollary for the dimension of $V(\mu)$ and explaining how to get explicit character values out of the formula. Applications of the Weyl Character Formula to $\mathsf{sl}_2(\mathbf{C})$ and $\mathsf{sl}_3(\mathbf{C})$ are given in Exercises 4.4 and 4.5 We mention that Kostant's Multiplicity Formula (see for instance [3, §8.2]) is also a quick corollary.

Character values and Weyl's Dimension Formula. Let G be the connected and simply connected Lie group with Lie algebra L. The Lie algebra homomorphism $L \to \operatorname{End}(V(\mu))$ induces a representation of G acting on $V(\mu)$. Fix $h \in H^\star$ and consider the one-parameter semigroup $c \mapsto \exp(ch) \in G$. By Exercise 4.1, the character value $\chi_{V(\mu)}(\exp ch)$ of $\chi_{V(\mu)}$ on $\exp ch \in G$ is obtained by replacing each formal exponential $\mathrm{e}(\alpha)$ with $e^{c\alpha(h)}$ in Weyl's Character Formula. (There is no abuse of notation here: $\chi_{V(\mu)}$ is the formal character; $\chi_{V(\mu)}(\exp ch)$ is this evaluation.) Hence, using the definition of the action of Weyl group on formal exponentials by $w \cdot e(\alpha) = e(w \cdot \alpha)$, we have

$$\chi_{V(\mu)}(\exp ch) = \frac{\sum_{w \in W} \operatorname{sgn}(w)\, e^{c(w\cdot(\mu+\delta))(h)}}{\sum_{w \in W} \operatorname{sgn}(w)\, e^{c(w\cdot\delta)(h)}}. \tag{4.1}$$

To obtain $\dim V(\mu)$ we would like to set $c = 0$, but this is not easy to do, even formally, because both numerator and denominator vanish at $c = 0$. In fact, we will see below that the order of vanishing is $|\Phi^+|$, which perhaps explains why all proofs of the dimension formula are a little indirect. Here we exploit that, by Exercise 4.3(iii), the denominator in Weyl's Character Formula is $\prod_{\alpha\in\Phi^+}\bigl(\mathrm{e}(\frac{1}{2}\alpha) - \mathrm{e}(-\frac{1}{2}\alpha)\bigr)$ which becomes $\prod_{\alpha\in\Phi^+}\bigl(e^{\frac{1}{2}c\alpha(h)} - e^{-\frac{1}{2}c\alpha(h)}\bigr)$ when evaluated at $h \in H$. Hence

$$\sum_{w\in W} \operatorname{sgn}(w)\, e^{c(w\cdot\delta)(h)} = \prod_{\alpha\in\Phi^+}\bigl(e^{\frac{1}{2}c\alpha(h)} - e^{-\frac{1}{2}c\alpha(h)}\bigr). \tag{4.2}$$

Using this, we show that in the very special case when $h = t_\delta$ (recall that t_δ is the element of H dual to $\delta \in H^\star$), the numerator of (4.1) factors in a similar way to the factorization $\prod_{\alpha\in\Phi^+}\bigl(e^{\frac{1}{2}c\alpha(h)} - e^{-\frac{1}{2}c\alpha(h)}\bigr)$ of the denominator just seen. Indeed,

$$\bigl(w\cdot(\mu+\delta)\bigr)(t_\delta) = \bigl(w\cdot(\mu+\delta), \delta\bigr) = \bigl(\mu+\delta, w\cdot\delta\bigr) = (w\cdot\delta)(t_{\mu+\delta})$$

and so

$$\begin{aligned}\sum_{w\in W}\operatorname{sgn}(w)e^{c(w\cdot(\mu+\delta))(t_\delta)} &= \sum_{w\in W}\operatorname{sgn}(w)e^{c(\alpha)(t_{\mu+\delta})}\\ &= \prod_{\alpha\in\Phi^+}\bigl(\mathrm{e}^{\frac{1}{2}c(\alpha)(t_{\mu+\delta})} - \mathrm{e}^{-\frac{1}{2}c(\alpha)(t_{\mu+\delta})}\bigr)\\ &= \prod_{\alpha\in\Phi^+}\bigl(\mathrm{e}^{\frac{1}{2}c(\alpha,\mu+\delta})-\mathrm{e}^{-\frac{1}{2}c(\alpha,\mu+\delta)}\bigr)\end{aligned}$$

where we used (4.2) for the second equality. Therefore taking $\mu = \varnothing$ in (4.2), or equivalently using the equation for the denominator just before (4.1), we obtain

$$\chi_{V(\mu)}(\exp ct_\delta) = \prod_{\alpha\in\Phi^+}\frac{\mathrm{e}^{\frac{1}{2}c(\alpha,\mu+\delta})-\mathrm{e}^{-\frac{1}{2}c(\alpha,\mu+\delta)}}{\mathrm{e}^{\frac{1}{2}c(\alpha,\delta)}-\mathrm{e}^{-\frac{1}{2}c(\alpha,\delta)}}$$

Now using that the factors in numerator and denominator are $c(\alpha,\mu+\delta)+\mathrm{O}(c^2)$ and $c(\alpha,\delta)+\mathrm{O}(c^2)$, their quotient has limit $(\alpha,\mu+\delta)/(\alpha,\delta)$ as $c \to 0$. It is clear that $\exp ct_\delta \to 1 \in G$ as $c \to 0$. Therefore taking the limit $c \to 0$ in the displayed equation above we get

$$\dim V(\mu) = \prod_{\alpha\in\Phi^+}\frac{(\alpha,\mu+\delta)}{(\alpha,\delta)} \tag{4.3}$$

which is Weyl's Dimension Formula.

5. Proof of the Weyl Character Formula

The following proof is adapted from Igusa's notes [5]. For calculations it will be convenient to extend $\mathbf{Q}[\Lambda]$ to a larger ring $\mathbf{Q}[\frac{1}{2}\Lambda]$ by adjoining a square root $e(\frac{1}{2}\alpha)$ for each $\alpha \in \Phi$. We then complete $\mathbf{Q}[\frac{1}{2}\Lambda]$ to the algebra $\mathbf{Q}[[\frac{1}{2}\Lambda]]$ of formal power series generated by the $\mathrm{e}(\frac{1}{2}\lambda)$ for $\lambda \in \Lambda$. For example, in $\mathbf{Q}[[\frac{1}{2}\Lambda]]$ we have $\sum_{s=0}^{\infty} \mathrm{e}(\lambda)^s = \frac{1}{1-\mathrm{e}(\lambda)}$.

We shall also need the *Laplacian operator* $\Delta : \mathbf{Q}[[\frac{1}{2}\Lambda]] \to \mathbf{Q}[[\frac{1}{2}\Lambda]]$, defined by $\Delta\big(\mathrm{e}(\lambda)\big) = ||\lambda||^2 e(\lambda)$ for $\lambda \in \frac{1}{2}\Lambda$, and the bilinear form $\{\ ,\ \}$ on $\mathbf{Q}[[\frac{1}{2}\lambda]]$ defined by

$$\{\mathrm{e}(\lambda), e(\mu)\} = (\lambda,\mu)\mathrm{e}(\lambda+\mu) \quad \text{for } \lambda,\mu \in \tfrac{1}{2}\Lambda.$$

See Exercise 4.3(i) and (iv) for some motivation for Δ. These gadgets are related by the following lemma.

Lemma 5.1. *Let $f, g \in \mathbf{Q}[[\frac{1}{2}\Lambda]]$. Then*

$$\Delta(fg) = f\Delta(g) + \Delta(f)g + 2\{f,g\}.$$

Proof. By linearity it is sufficient to prove the lemma when $f = \mathrm{e}(\lambda)$ and $g = \mathrm{e}(\mu)$ for some $\lambda, \mu \in \frac{1}{2}\Lambda$. In this case it states that

$$||\lambda+\mu||^2\mathrm{e}(\lambda+\mu) = \mathrm{e}(\lambda)||\mu||^2\mathrm{e}(\mu) + ||\lambda^2||\mathrm{e}(\lambda)\mathrm{e}(\mu) + 2(\lambda,\mu)\mathrm{e}(\lambda+\mu)$$

which is obvious. □

Proof of Weyl Character Formula. Let Q denote the denominator in the Weyl Character Formula. We begin the proof with Freudenthal's formula in the form

$$\begin{aligned}\big(||\mu+\delta||^2 - &||\delta||^2\big)n(\mu)_\lambda \\ &= \big(||\lambda||^2 + (\lambda,2\delta)\big)n(\mu)_\lambda + 2\sum_{\alpha\in\Phi^+}\sum_{m=1}^{\infty}(\lambda+m\alpha,\alpha)n(\mu)_{\lambda+m\alpha}.\end{aligned}$$

Multiply both sides by $\mathrm{e}(\lambda)$ and sum over all $\lambda \in \Lambda$ to get

$$\big(||\mu+\delta||^2 - ||\delta||^2\big)\chi_V = \Delta(\chi_V) + \sum_{\lambda\in\Lambda}(\lambda,2\delta)n(\mu)_\lambda\mathrm{e}(\lambda) + X \tag{5.1}$$

where

$$\begin{aligned} X &= 2\sum_{\lambda\in\Lambda}\sum_{\alpha\in\Phi^+}\sum_{m=1}^{\infty}(\lambda+m\alpha,\alpha)n(\mu)_{\lambda+m\alpha}\mathrm{e}(\lambda) \\ &= 2\sum_{\lambda\in\Lambda}\sum_{\alpha\in\Phi^+}\sum_{m=1}^{\infty}(\lambda,\alpha)n(\mu)_{\lambda}\mathrm{e}(\lambda-m\alpha) \\ &= 2\sum_{\lambda\in\Lambda}\sum_{\alpha\in\Phi^+}(\lambda,\alpha)\frac{n(\mu)_\lambda\mathrm{e}(\lambda)}{\mathrm{e}(\alpha)-1}. \end{aligned}$$

Now multiply through by Q and replace 2δ with $\sum_{\alpha\in\Phi^+}\alpha$ to combine the second two summands on the right-hand side of (5.1). This gives

$$\big(||\mu+\delta||^2 - ||\delta||^2\big)Q\chi_V = Q\Delta(\chi_V) + Q\sum_{\lambda\in\Lambda}\sum_{\alpha\in\Phi^+}(\lambda,\alpha)n(\mu)_\lambda \mathrm{e}(\lambda)\frac{\mathrm{e}(\alpha)+1}{\mathrm{e}(\alpha)-1}.$$

Since $Q\chi_V$ is antisymmetric, it follows from Exercise 4.3(i) that $Q\chi_V = \sum_{w\in W}\mathrm{sgn}(w)w\cdot e(\mu+\delta)$ if and only if $\Delta(Q\chi_V) = ||\mu+\delta||^2 Q\chi_V$. Again by this exercise, $\Delta(Q) = ||\delta||^2 Q$. Hence it is sufficient to prove

$$\Delta(Q\chi_V) - \Delta(Q)\chi_V - Q\Delta(\chi_V) = Q\sum_{\lambda\in\Lambda}\sum_{\alpha\in\Phi^+}(\lambda,\alpha)n(\mu)_\lambda \mathrm{e}(\lambda)\frac{\mathrm{e}(\alpha)+1}{\mathrm{e}(\alpha)-1}. \tag{5.2}$$

By Lemma 5.1, the left-hand side in (5.2) is $2\{Q,\chi_V\}$. So finally, it is sufficient to prove that

$$2\big\{Q, \sum_{\lambda\in\Lambda} n(\mu)_\lambda \mathrm{e}(\lambda)\big\} = Q\sum_{\alpha\in\Phi^+}\frac{\mathrm{e}(\alpha)+1}{\mathrm{e}(\alpha)-1}\sum_{\lambda\in\Lambda}(\lambda,\alpha)n(\mu)_\lambda\mathrm{e}(\lambda)$$

which, by linearity, follows from the relation

$$2\big\{Q,\mathrm{e}(\nu)\big\} = Q\sum_{\alpha\in\Phi^+}\frac{\mathrm{e}(\alpha)+1}{\mathrm{e}(\alpha)-1}(\nu,\alpha)e(\nu) \quad \text{for } \nu\in\Lambda,$$

proved in Exercise 5.2 below. □

6. Applications to Type A and Type C

Fix $n\in\mathbf{N}$ and for $1\le i,j\le n$ let $e_{(i,j)}$ be the $n\times n$ matrix whose single non-zero entry is a 1 in position (i,j). Let D be the abelian Lie algebra of diagonal $n\times n$ matrices and let $\varepsilon_i\in D^\star$ be defined by $\varepsilon_i(d) = d_{(i,i)}$. The basic commutator relation

$$[e_{(i,j)}, e_{(k,m)}] = [j=k]e_{(i,m)} - [i=m]e_{(k,j)} \tag{6.1}$$

will often be useful in this section; recall that $[j=k]$ is the Iverson bracket, equal to 1 if $j=k$ and 0 otherwise, and similarly for $[i=m]$.

Type A. Let H be the Cartan subalgebra of the special linear Lie algebra sl_n consisting of all trace 0 diagonal matrices. By restricting the elements $\varepsilon_i\in D^\star$ just defined we see that $H^\star = \langle\varepsilon_1,\ldots,\varepsilon_n\rangle_{\mathbf{C}}$. (Note this is a spanning set, not a basis, because $\varepsilon_1+\cdots+\varepsilon_n = 0$, and correspondingly the Lie rank ℓ is $n-1$, not n.)

Root space decomposition. From (6.1) we get

$$[e_{(k,k)}, e_{(i,j)}] = \big([i=k]-[j=k]\big)e_{(i,j)} = (\varepsilon_i - \varepsilon_j)e_{(i,j)}. \tag{6.2}$$

Thus the $e_{(i,j)}$ for $i\ne j$ span the root spaces and we may choose the positive roots to be

$$\Phi^+ = \{\varepsilon_i - \varepsilon_j : 1\le i<j\le n\}$$

The root system Φ has $\varepsilon_i - \varepsilon_{i+1}$ for $1\le i\le n$ as a base so that, in the notation of §2, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$.

Weight lattice. The dual elements in the Cartan subalgebra to α_i, satisfying the fundamental relation (2.5) that $\alpha_i(h_j) = \langle \alpha_i, \alpha_j \rangle$ are $h_i = e_{(i,i)} - e_{(i+1,i+1)}$ and the fundamental dominant weights in (2.6) are $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ for $1 \leq i < n$. It is easily checked that $\omega_i(h_j) = [i = j]$; the case $n = 3$ was seen in Exercise 2.8. Thus the weight lattice is $\Lambda = \langle \varepsilon_1, \ldots, \varepsilon_n \rangle_{\mathbf{Z}}$ and the dominant integral weights in $\langle \omega_1, \ldots, \omega_{n-1} \rangle_{\mathbf{N}_0}$ are partitions, written allowing zero parts and having at most $n-1$ non-zero parts.

Weyl group. It is easily shown that the reflection $S_{\varepsilon_i - \varepsilon_j}$ defined in §4 acts on $\mathbf{R}^n$ by swapping positions i and j in vectors. Therefore the Weyl group of type A_{n-1} is the symmetric group $\mathfrak{S}_n$ and the sign function as defined before Definition 4.1 in terms of simple reflections, is the usual sign function the symmetric group, which to avoid ambiguity we denote by $\mathrm{sgn}_{\mathfrak{S}_n}$.

Weyl's Character Formula and antisymmetric polynomials. By Exercise 6.1 we have $\delta = \sum_{i=1}^{n-1} (n-i)\varepsilon_i$ which we write as the partition $(n-1, \ldots, 1, 0)$. We may identify the subring of $\mathbf{Q}[\Lambda]$ generated by the $e(\varepsilon_i)$ with the polynomial ring $\mathbf{Z}[x_1, \ldots, x_n]$ via the isomorphism $\mathrm{e}(\varepsilon_i) \mapsto x_i$ for $1 \leq i \leq n$. (Strictly speaking we are working in the quotient of this ring by $x_1 \ldots x_n$ since $\varepsilon_1 + \cdots + \varepsilon_n = 0$, but since ε_n does not appear in a fundamental dominant weight, and correspondingly our partitions have at most $n-1$ parts, we can ignore this point.) After this identification, the Weyl group $\mathfrak{S}_n$ acts on $\mathbf{Z}[x_1, \ldots, x_n]$ by polynomial extension of $w \cdot x_i = x_{w(i)}$. Weyl's Character Formula (see Theorem 4.2) becomes

$$\chi_{V(\mu)} = \frac{\sum_{w \in \mathfrak{S}_n} \mathrm{sgn}_{\mathfrak{S}_n}(w) w \cdot x_1^{\mu_1 + (n-1)} x_2^{\mu_2 + (n-2)} \cdots x_{n-1}^{\mu_{n-1}+1} x_n^{\mu_n}}{\sum_{w \in \mathfrak{S}_n} \mathrm{sgn}_{\mathfrak{S}_n}(w) w \cdot x_1^{n-1} x_2^{n-2} \cdots x_{n-1}}$$

$$\begin{vmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{vmatrix}$$

The denominator is the Vandermonde determinant: see the margin for the case $n = 3$. More generally, observe that the numerator is the determinant

$$\begin{vmatrix} x_1^{\mu_1+(n-1)} & x_1^{\mu_2+(n-2)} & \ldots & x_1^{\mu_n} \\ x_2^{\mu_1+(n-1)} & x_2^{\mu_2+(n-2)} & \ldots & x_2^{\mu_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\mu_1+(n-1)} & x_n^{\mu_2+(n-2)} & \ldots & x_n^{\mu_n} \end{vmatrix}$$

Therefore we have an explicit formula for the formal character $\chi_{V(\mu)}$ as a quotient of two antisymmetric polynomials:

$$\chi_{V(\mu)} = \frac{\left| x_i^{\mu_j + (n-j)} \right|_{1 \leq i \leq j \leq n}}{\left| x_i^{n-j} \right|_{1 \leq i \leq j \leq n}}$$

where the denominator is the Vandermonde determinant $\prod_{1 \leq i < j \leq n} (x_i - x_j)$.

Schur polynomials. Since $\chi_{V(\mu)}$ is a quotient of two antisymmetric polynomials it is symmetric, as expected because it is invariant under the Weyl group $\mathfrak{S}_n$. In fact it is the Schur polynomial $s_\mu(x_1, \ldots, x_n)$ enumerating semistandard tableaux of shape μ with entries from $\{1, \ldots, n\}$ by their weight (i.e. their multiset of entries): see [9, Ch. 7] for much more on symmetric functions. As a challenge, the reader is invited to deduce Stanley's Hook Content Formula (see [9, Theorem 7.21.2]) by specializing the formula above by $x_i \mapsto q^{i-1}$. This result may be regarded a greatly refined version of Weyl's Dimension Formula, for the Type A case.

Type C. Write elements of $\mathbf{C}^{2n}$ as pairs (v, w) with $v, w \in \mathbf{C}^n$. We define the symplectic Lie algebra $\mathsf{sp}_{2n}(\mathbf{C})$ of rank n using the symplectic form

$$\big((v, w), (v', w')\big) = v \cdot w' - w \cdot v'$$

where $\cdot$ is the usual dot product. The matrix of this form in the canonical basis of $\mathbf{C}^{2n}$ is J, as shown in the margin. Hence,

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

$$\mathsf{sp}_{2n}(\mathbf{C}) = \big\{z \in \mathsf{gl}_{2n}(\mathbf{C}) : z^t J + Jz = 0\big\}$$

or equivalently, by Exercise 6.2,

$$\mathsf{sp}_{2n}(\mathbf{C}) = \left\{ \begin{pmatrix} a & b \\ c & a^{\mathrm{tr}} \end{pmatrix} : b = b^{\mathrm{tr}}, c = c^{\mathrm{tr}} \right\}.$$

(Part of this can be seen without calculation: since the subspace $\langle v_1, \ldots, v_n\rangle$ is totally isotropic, and is perfectly paired with $\langle v_{n+1}, \ldots, v_{2n}\rangle$ by the form, the top-left block can be freely chosen and determines the bottom right block. But it's probably simpler just to do the calculation then pursue this argument to work out all the signs.) In particular, $\mathsf{sp}_{2n}(\mathbf{C})$ has the diagonal matrices with entries $d_1, \ldots, d_n, -d_1, \ldots, -d_n$ as a Cartan subalgebra.

Root space decomposition. Writing $\bar{i}$ for $i + n$, it follows from (6.2) that

$$\begin{aligned}[][e_{(k,k)} - e_{(\bar{k},\bar{k})}, e_{(i,j)} + e_{(\bar{j},\bar{i})}] &= ([i = k] - [j = k])e_{(i,j)} - ([\bar{j} = \bar{k}] - [\bar{i} = \bar{k}])e_{(\bar{j},\bar{i})} \\ &= ([i = k] - [j = k])(e_{(i,j)} + e_{(\bar{j},\bar{i})})\end{aligned}$$

and hence $e_{(i,j)} + e_{(\bar{j},\bar{i})}$ is in the $\varepsilon_i - \varepsilon_j$ root space. Similarly we have

$$[e_{(k,k)} - e_{(\bar{k},\bar{k})}, e_{(i,\bar{j})}] = ([i = k] + [\bar{j} = \bar{k}])e_{(i,\bar{j})}$$

and so $e_{(i,\bar{j})}$ is in the $\varepsilon_i + \varepsilon_j$ root space; note that taking $j = i$ (as we may since b and c are symmetric), this gives the $2\varepsilon_i$ root spaces. We choose the positive roots Φ^+ so that

$$\Phi^+ = \{\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le n\} \cup \{2\varepsilon_i : 1 \le i \le n\} \tag{6.3}$$

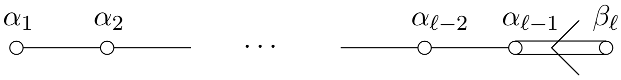
and get the Type C root system having

$$\varepsilon_1 - \varepsilon_2, \ldots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n$$

as a base so that, in the notation of §2, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \le i < n$ and $\alpha_n = 2\varepsilon_n$. Thus

$$h_i = e_{(i,i)} - e_{(i+1,i+1)} - e_{(\bar{i},\bar{i})} + e_{(\overline{i+1},\overline{i+1})} \tag{6.4}$$

for $1 \le i < n$ and $h_n = e_{(n,n)} - e_{(\bar{n},\bar{n})}$ The values of $-\langle \alpha_i, \alpha_j\rangle$ are shown by the number of edges in the Dynkin diagram below; the arrow indicates that $\beta_\ell = 2\varepsilon_\ell$ is the longer root.



See Exercise 6.4 for the special case of $\mathsf{sp}_4(\mathbf{C})$.

Weight lattice. Exactly as in the Type A case, the fundamental dominant weights in (2.6) for $1 \le i < n$ are $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ and clearly $(\varepsilon_1 + \cdots + \varepsilon_{n-1} + \varepsilon_n)(h_i) = [i = n]$ so we now have $\omega_n = \varepsilon_1 + \cdots + \varepsilon_n$. Therefore we may identify dominant integral weights with partitions.

Weyl group. Again the reflection $S_{\varepsilon_i - \varepsilon_j}$ acts on $\mathbf{R}^n$ by swapping positions i and j in roots, and it is clear that $S_{2\varepsilon_n}$ acts by negating position n. Therefore the Weyl group of type C_n is the hyperoctahedral group $C_2 \wr \mathfrak{S}_n$ acting as the group of all signed permutation matrices on $\mathbf{R}^n$. This is the semidirect product $(C_2 \times \cdots \times C_2) \rtimes \mathfrak{S}_n$ with $\mathfrak{S}_n$ acting on the base group by position permutation. We write its elements as $(a_1, \ldots, a_n; w)$ where $a_i \in \{1, -1\}$ for each i and $w \in \mathfrak{S}_n$.

Weyl's Character Formula and antisymmetric polynomials. By Exercise 6.3 we have $\delta = \sum_{i=1}^n (n+1-i)\varepsilon_i$. We identify $\mathbf{Q}[\Lambda]$ with the polynomial ring $\mathbf{Z}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ via the isomorphism $\mathrm{e}(\varepsilon_i) \mapsto x_i$ and $\mathrm{e}(-\varepsilon_i) \mapsto x_i^{-1}$ for $1 \le i \le n$. After this identification, the Weyl group $C_2 \wr \mathfrak{S}_n$ acts on $\mathbf{Z}[x_1, \ldots, x_n]$ by algebra extension of

$$(a_1, \ldots, a_n; w) \cdot x_i = \begin{cases} x_{w(i)} & \text{if } a_{w(i)} = 1 \\ x_{w(i)}^{-1} & \text{if } a_{w(i)} = -1. \end{cases}$$

Since the sign function is a character of $C_2 \wr \mathfrak{S}_n$ and, as in the Type A case, $\operatorname{sgn}(1, \ldots, 1; w) = \operatorname{sgn}_{\mathfrak{S}_n}(w)$ we have

$$\operatorname{sgn}(a_1, \ldots, a_n; w) = (-1)^{a_1} \cdots (-1)^{a_n} \operatorname{sgn}_{\mathfrak{S}_n}(w).$$

Weyl's Character Formula (see Theorem 4.2) becomes

$$\begin{aligned} \chi_{V(\mu)} &= \frac{\sum_{v \in C_2 \wr \mathfrak{S}_n} \operatorname{sgn}(v) v \cdot x_1^{\mu_1+n} \cdots x_n^{\mu_n+1}}{\sum_{v \in C_2 \wr \mathfrak{S}_n} \operatorname{sgn}(v) v \cdot x_1^n \cdots x_n} \\ &= \frac{\sum_{w \in \mathfrak{S}_n} \operatorname{sgn}_{\mathfrak{S}_n}(w) w \cdot (x_1^{\mu_1+n} - x_1^{-\mu_1-n}) \cdots (x_n^{\mu_n+1} - x_n^{\mu_n-1})}{\sum_{w \in C_2 \wr \mathfrak{S}_n} \operatorname{sgn}_{\mathfrak{S}_n}(w) w \cdot (x_1^n - x_1^{-n}) \cdots (x_n - x_n^{-1})} \end{aligned}$$

Note the minus sign appears in each factor because of the sign from the base group. As in the Type A case, both numerator and denominator are determinants: the numerator is

$$\begin{vmatrix} x_1^{\mu_1+n} - x_1^{-\mu_1+n} & x_1^{\mu_2+(n-1)} - x_1^{-\mu_2-n+1} & \ldots & x_1^{\mu_n+1} - x_1^{-\mu_n-1} \\ x_2^{\mu_1+n} - x_2^{-\mu_1-n} & x_2^{\mu_2+(n-1)} - x_2^{-\mu_2-n+1} & \ldots & x_2^{\mu_n+1} - x_2^{-\mu_n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\mu_1+n} - x_n^{-\mu_1-n} & x_n^{\mu_2+(n-1)} - x_n^{-\mu_2-n+1} & \ldots & x_n^{\mu_n+1} - x_n^{-\mu_n-1} \end{vmatrix}$$

and there is a similar expression for the denominator, giving

$$\chi_{V(\mu)} = \frac{\left|x_i^{\mu_j+(n+1-j)} - x_i^{-\mu_j-(n+1-j)}\right|_{1\le i\le j\le n}}{\left|x_i^{n+1-j} - x_i^{-(n+1-j)}\right|_{1\le i\le j\le n}}.$$

Further directions. A recommended exercise is to repeat the analysis of this section for orthogonal groups (Types B and D). For analogues of Stanley's Hook Content Formula in Type B, C and D see [1]. It would be very interesting to have an analogue of James' abacus [7, Ch. 4] or Loehr's labelled abacus [8] for the symplectic case.

Exercises

Exercise 1.1. Let $E = \langle u, v\rangle$ be the natural 2-dimensional $\mathsf{sl}_2(\mathbf{C})$-module. Show that $\mathrm{Sym}^d E$ is irreducible for each $d \in \mathbf{N}$.

Exercise 1.2. Let V be a finite-dimensional $\mathsf{sl}_2(\mathbf{C})$-module.

(i) Show that V contains an h-eigenvector v such that $e \cdot v = 0$.
(ii) Show that the submodule of V generated by V is d-dimensional if and only if $h \cdot v = (d-1)v$.
(iii) Deduce that any irreducible $\mathsf{sl}_2(\mathbf{C})$-module is isomorphic to $\mathrm{Sym}^d E$ for some $d \in \mathbf{N}_0$.

Exercise 2.1. Show that a Cartan subalgebra (as defined in Section 2) is abelian.

Solution. Given $h,\ k \in H$, we can write k as a sum of $\mathrm{ad}\, h$ eigenvectors, say $k = k_0 + \sum_{i=1}^n k_i$ where $(\mathrm{ad}\, h)k_0 = 0$ and $(\mathrm{ad}\, h)k_i = \lambda_i k_i$. Hence $(\mathrm{ad}\, h)^r k = \sum_{i=1}^n \lambda_i^r k_i$. A useful linear algebra lemma shows that all the k_i are in the Lie subalgebra of H generated by h and k. Now $[h, k_i] = \lambda_i k_i$ and so $(\mathrm{ad}\, k_i)^2 x = [k_i, [k_i, x]] = [k_i, -\lambda_i k_i] = 0$; since $k_i \in H$, $\mathrm{ad}\, k_i$ is diagonalizable, and so we must have $(\mathrm{ad}\, k_i)x = 0$. Hence $[h, k] = 0$.

Exercise 2.2. The aim of this exercise is to show that if H is a Cartan subalgebra of L then H is self-centralizing.

(i) Show that L_0 is nilpotent. [*Hint:* use Engel's theorem and the abstract Jordan decomposition.]
(ii) Show that there is a basis of L_0 in which all $\mathrm{ad}\, x : L \to L$ for $x \in L_0$ are represented by upper-triangular matrices.
(iii) Show that if $x \in L_0$ and $\mathrm{ad}\, x : L \to L$ is nilpotent then $\mathrm{Tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y) = 0$ for all $y \in L_0$. Deduce that $x = 0$.
(iv) Deduce that every element of L_0 is semisimple and hence show that $L_0 = H$.

Exercise 2.3. Let V be a complex vector space. Show that if x and $y \in \mathsf{gl}(V)$ are such that $[x, y]$ commutes with x then $[x, y]$ is nilpotent. [*Hint:* there is a quick solution using Lie's Theorem. For an *ad-hoc* proof (which then allows this exercise to be used as part of a proof of Lie's Theorem) first show that $\mathrm{Tr}[x, y]^n = 0$ for all $n \in \mathbf{N}$.]

Exercise 2.4. Suppose that L is simple. Show that Φ spans $H^\star$ by using $[L_\alpha, L_{-\alpha}] = \langle t_\alpha \rangle = \langle h_\alpha \rangle$ and $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ to deduce that the subspace $\langle t_\alpha : \alpha \in \Phi \rangle \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ is an ideal of L.

Exercise 2.5. Let α and β be non-perpendicular roots in a root system. Use the fundamental relation (2.5) to find the possible angles between α and β and the possible values of $||\alpha||/||\beta||$.

Exercise 2.6. Find the Killing form of $\mathsf{sl}_2(\mathbf{C})$ with respect to the basis e, f, h and hence calculate $||\alpha||^2$ where α is the unique root of $\mathsf{sl}_2(\mathbf{C})$. (In practice the previous exercise always gives enough information, so this calculation is unnecessary. For example, this remark applies to Freudenthal's formula, since n_λ is expressed as a quotient of norms, and to Exercise **??**, for the same reason.)

Exercise 2.7. Let V be a finite-dimensional L-module and let $v \in V$ be a highest-weight vector. Show that the submodule of L generated by v is irreducible.

Exercise 2.8. Let H be the Cartan subalgebra of diagonal matrices in $\mathsf{sl}_3(\mathbf{C})$. For $i \in \{1, 2, 3\}$, let $\varepsilon_i : H \to \mathbf{C}$ be the function sending $\mathrm{diag}(a_1, a_2, a_3)$ to a_i. Let $\alpha = \varepsilon_1 - \varepsilon_2$ and let $\beta = \varepsilon_2 - \varepsilon_3$.

(i) Show that $\{\alpha, \beta\}$ is a base for the root system Φ.
(ii) Show that $||\alpha|| = ||\beta||$ and that the angle between α and β is $2\pi/3$.
(iii) Find the fundamental dominant integral weights ω_1, ω_2 corresponding to this base in terms of α and β.
(iv) Show that $\omega_1 = \varepsilon_1$ and $\omega_2 = \varepsilon_1 + \varepsilon_2$. (Since $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ other, equivalent, expressions for ω_1 and ω_2 are also possible.)
(iv) Express the highest weight of the natural, dual natural and adjoint representations of $\mathsf{sl}_3(\mathbf{C})$ as $\mathbf{Z}$-linear combinations of ω_1 and ω_2.
(v) Confirm that the weight lattice $\Lambda = \langle \varepsilon_1, \varepsilon_1 + \varepsilon_2 \rangle_{\mathbf{Z}}$ and cone of dominant integral weights are as shown in Figure 1 below.

Exercise 3.1. Recall from (3.2) that $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. We have chosen a base $B = \{\alpha_1, \ldots, \alpha_\ell\}$ for Φ.

(i) Show that if $\beta \in \Phi^+$ and $\beta \neq \alpha_i$ then $S_{\alpha_i}(\beta) \in \Phi^+$
(ii) Show that $S_{\alpha_i}(\delta) = \delta - \alpha_i$ for all i.
(iii) Show that $\delta = \omega_1 + \cdots + \omega_\ell$ and deduce that $\delta \in \Lambda$.

In (iii) recall that, by (2.6), the ω_i are the elements of the weight lattice Λ dual to the chosen basis $h_1, \ldots, h_\ell$ of the Cartan subalgebra.

Solution. (i) Since $\beta \neq \alpha_i$ and $k\alpha_i$ is a root if and only if $k \in \{+1, -1\}$ (see, for example, [2, Proposition 10.9]), there exists j such that α_j appears with a strictly positive coefficient in the expression for β as a $\mathbf{Z}$-linear combination of $\alpha_1, \ldots, \alpha_n$. Now α_j has the same coefficient in

$$S_{\alpha_i}(\beta) = \beta - \langle \beta, \alpha_i \rangle \alpha_i,$$

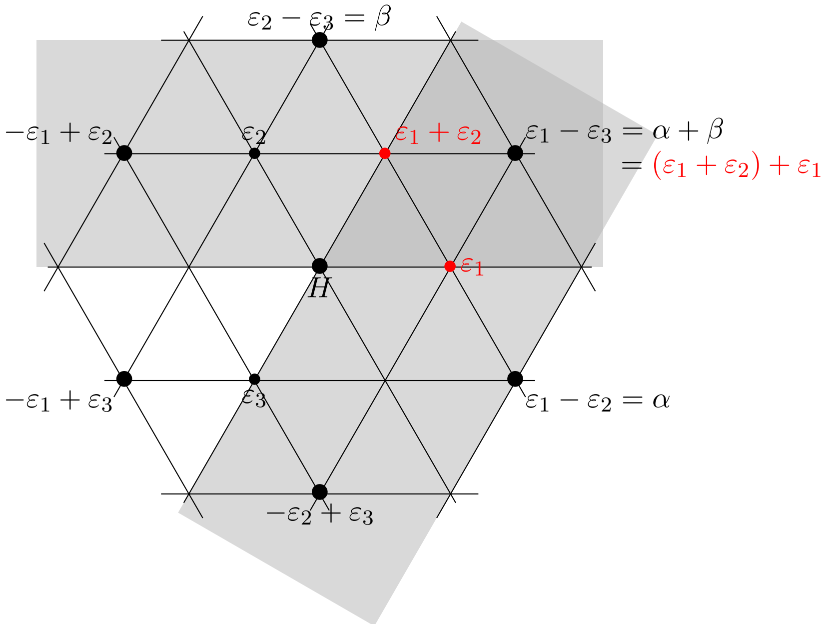


FIGURE 1. Weight spaces for $\mathsf{sl}_3(\mathbf{C})$ representations, showing the weights ε_1, $\varepsilon_1+\varepsilon_2$ (dominant) and $\varepsilon_2, \varepsilon_3$ and the elements e_{ij} spanning the root spaces for $\varepsilon_i - \varepsilon_j$. The positive half-space for α and β are shaded: their intersection is the cone of dominant integral weights $\{a\varepsilon_1 + b(\varepsilon_1+\varepsilon_2) : a, b \in \mathbf{N}_0\}$.

and so it follows that $S_{\alpha_i}(\beta) \in \Phi^+$.

(ii) Since S_{α_i} permutes $\Phi^+ \backslash \{\alpha_i\}$ and $S_{\alpha_i}(\alpha_i) = -\alpha_i$, we have

$$S_{\alpha_i}(\delta) = \tfrac{1}{2} \sum_{\beta \in \Phi} S_{\alpha_i}(\beta) = \tfrac{1}{2} \sum_{\beta \in \Phi} S_{\alpha_i}(\beta) - \alpha_i = \delta - \alpha_i$$

as required.

(iii) By the definition in (2.6), $\langle \alpha_i, \omega_j \rangle = 0$ if $i \neq j$ and $\langle \alpha_i, \omega_j \rangle = 1$. Hence

$$S_{\alpha_j}\Big(\sum_{i=1}^{\ell} \omega_i\Big) = \sum_{i=1}^{\ell} \omega_i - \omega_j + S_{\alpha_j}(\omega_j) = \sum_{i=1}^{\ell} \omega_i - \omega_j + \omega_j - \alpha_j = \sum_{i=1}^{\ell} \omega_i - \alpha_j.$$

Hence by (ii), $-\delta + \sum_{i=1}^{\ell} \omega_i$ is invariant under the generators $S_{\alpha_1}, \ldots, S_{\alpha_\ell}$ of W. Hence $\delta = \sum_{i=1}^{\ell} \omega_i \in \Lambda$.

Exercise 3.2. Let $B : V \to V$ be a non-degenerate symmetric bilinear form on an n-dimensional vector space V. Suppose that $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are dual bases for V, so

$$B(x_i, y_j) = [i = j]$$

Let $\theta \in V^\star$ and let t_θ be the unique element such that $B(t_\theta, v) = \theta(v)$ for all $v \in V$. Let $v \in V$.

(i) Show that $v = \sum_{i=1}^{n} B(x_i, v) y_i = \sum_{j=1}^{n} B(v, y_j) x_j$.

(ii) Hence show that $B(t_\theta, t_\theta) = \sum_{k=1}^n \theta(x_k)\theta(y_k)$.

Solution. (i) For each j we have $B\big(\sum_{i=1}^n B(x_i, v)y_i, x_j\big) = B(x_j, v)$, hence $B\big(-v + \sum_{i=1}^n B(x_i, v)y_i, x_j\big) = 0$ for all j. Since $x_1, \ldots, x_n$ is a basis of V and B is non-degenerate, it follows that $v = \sum_{i=1}^n B(x_i, v)y_i, x_j$, as required. Similarly one finds that $v = \sum_{j=1}^n B(v, y_j)x_j$.

(ii) We have $t_\theta = \sum_{i=1}^n B(x_i, t_\theta)y_i$ and $t_\theta = \sum_{j=1}^n B(t_\theta, y_j)x_j$. Hence

$$(t_\theta, t_\theta) = \sum_{k=1}^n B(x_k, t_\theta)B(t_\theta, y_k) = \sum_{k=1}^n t_\theta(x_k)t_\theta(y_k)$$

as required.

Exercise 3.3. Prove Lemma 3.1. [*Hint:* Show that $\sum_{k=1}^n [x_k y_k, w] = \sum_{k=1}^n x_k[y_k w] + \sum_{k=1}^n [x_k w]y_k$ for $w \in L$, and then use Exercise 3.2(i) to express $[y_k, w]$ as a linear combination of $y_1, \ldots, y_n$ and $[x_k, w]$ as a linear combination of $x_1, \ldots, x_n$.]

Solution. Since $\mathcal{U}(L)$ is generated, as an algebra, by L, it is sufficient to prove that $[\sum_{k=1}^n x_k y_k, w] = 0$ for each $w \in L$. A routine calculation gives the result stated in the hint that

$$\sum_{k=1}^n [x_k y_k, w] = \sum_{k=1}^n x_k[y_k, w] + \sum_{k=1}^n [x_k, w]y_k.$$

By Exercise 3.2(i) we have $[y_k, w] = \sum_{i=1}^n \kappa(x_i, [y_k, w])y_i$ and $[x_k, w] = \sum_{j=1}^n \kappa([x_k, w], y_j)x_j$. Substituting we get

$$\sum_{k=1}^n [x_k y_k, w] = \sum_{k=1}^n \sum_{i=1}^n \kappa(x_i, [y_k, w])x_k y_i + \sum_{k=1}^n \sum_{j=1}^n \kappa([x_k, w], y_j)x_j y_k.$$

Now change the summation variables in the second sum and use the associativity of the Killing form to get

$$\begin{aligned}\sum_{k=1}^n [x_k y_k, w] &= \sum_{k=1}^n \sum_{i=1}^n \kappa(x_i, [y_k, w])x_k y_i + \sum_{i=1}^n \sum_{k=1}^n \kappa([x_i, w], y_k)x_k y_i \\ &= \sum_{k=1}^n \sum_{i=1}^n \big(-\kappa(x_i, [w, y_k]) + \kappa([x_i, w], y_k)\big)x_k y_i \\ &= 0\end{aligned}$$

as required.

Exercise 3.4. Take the notation from Lemma 3.3. Suppose that $\lambda(h_\alpha) \leq 0$.

(i) Deduce from (b) in Section 1 that if $U^{(i)}$ is a summand with *lowest* weight $(\lambda - b\alpha)(h_\alpha)$ where $b \in \mathbf{N}_0$, then $f_\alpha e_\alpha$ acts on $U^{(i)}_\lambda$ as the scalar $(b - \lambda(h_\alpha))(b + 1)$.

(ii) Show that the number of summands $U^{(i)}$ with lowest weight $(\lambda - b\alpha)(h_\alpha)$ is $n_{\lambda - b\alpha} - n_{\lambda-(b+1)\alpha}$.

(iii) Hence show that $f_\alpha e_\alpha$ acts on V_λ as the scalar $-\sum_{b=0}^{\infty} n_{\lambda-b\alpha}\langle \lambda - b\alpha, \alpha\rangle$, as claimed in the proof of Lemma 3.3.

Solution. (i) If $U^{(i)}$ has lowest weight $(\lambda - b\alpha)(h_\alpha)$ then $U^{(i)}$ has highest weight $-(\lambda - b\alpha)(h_\alpha)$. If $v \in U^{(i)}_\lambda$ then

$$h \cdot v = \lambda(h) = (-\lambda - b\alpha)(h_\alpha) - 2(b - \lambda(h_\alpha))$$

and so taking $c = b - \lambda(h_\alpha)$ in (b) in Section 1 gives

$$\begin{aligned} f \cdot e \cdot v &= (b - \lambda(h_\alpha))\big((-\lambda - b\alpha)(h_\alpha) - (b - \lambda(h_\alpha)) + 1\big)v \\ &= (b - \lambda(h_\alpha))(b+1)v \end{aligned}$$

as required. Now (ii) follows from (a) in Section 1, in the same way as (c) did, and (iii) is an immediate corollary of (i) and (ii).

Exercise 3.5. Let ω_1, ω_2 be the fundamental dominant weights for $\mathsf{sl}_3(\mathbf{C})$ (see Exercise 2.8). Use Freudenthal's Formula to determine the dimensions of the weight spaces for the $\mathsf{sl}_3(\mathbf{C})$-module with highest weight $2\omega_1 + \omega_2$.

Exercise 4.1. Let $\tau : L \to \mathsf{gl}(V)$ be a representation of L. Let G be the simply connected Lie group corresponding to L and let $\rho : G \to \mathrm{GL}(V)$ be the corresponding representation of G, as defined by

$$\rho(\exp x) = \exp\big(\tau(x)\big) \quad \text{for } x \in L.$$

(This defines ρ on a generating set for G.) Let $\lambda \in \Lambda$. Show that if $h \in H$ and $v \in V_\lambda$ then $\rho(\exp h)v = \exp\big(\lambda(h)\big)v$.

Exercise 4.2. Show that if V is an L-module then $\chi_V \in \mathbf{Q}[\Lambda]$ is symmetric in the sense of Definition 4.1.

Exercise 4.3. Let Λ_{dom} be the set of *strictly* dominant weights in Λ.

(i) Given $\lambda \in \Lambda$ define $a(\lambda) = \sum_{w \in W} \operatorname{sgn}(w)\, w \cdot \mathrm{e}(\lambda)$. Show that $\Delta\big(a(\lambda)\big) = ||\lambda||^2 a(\lambda)$ and deduce that $\{a(\lambda) : \lambda \in \Lambda_{\mathrm{dom}}\}$ is a $\mathbf{Z}$-basis of Δ-eigenvectors for the set of all antisymmetric elements of $\mathbf{Q}[\Lambda]$, in the sense of Definition 4.1.

(ii) Show that

$$\mathrm{e}(-\delta) \prod_{\alpha \in \Phi^+} \big(\mathrm{e}(\alpha) - 1\big) = \prod_{\alpha \in \Phi^+} \big(\mathrm{e}(\tfrac{1}{2}\alpha) - \mathrm{e}(-\tfrac{1}{2}\alpha)\big)$$

and that either side is antisymmetric.

(iii) Show that

$$\sum_{w \in W} \operatorname{sgn}(w)\, w \cdot \mathrm{e}(\delta) = \prod_{\alpha \in \Phi^+} \big(\mathrm{e}(\tfrac{1}{2}\alpha) - \mathrm{e}(-\tfrac{1}{2}\alpha)\big)$$

(iv) Prove that $f \in \mathbf{Q}[\tfrac{1}{2}\Lambda]$ is antisymmetric if and only if

$$f = g \prod_{\alpha \in \Phi^+} \big(\mathrm{e}(\tfrac{1}{2}\alpha) - \mathrm{e}(-\tfrac{1}{2}\alpha)\big)$$

for some symmetric g.

Solution. (i) Fix a total order on Λ refining the dominance order. Define the *degree* of an antisymmetric element f to be the greatest weight μ in this order such that $e(\mu)$ has a non-zero coefficient in f. If μ is the greatest weight of f then $\mu \in \Lambda_{\mathrm{dom}}$ and μ is acted on regularly by the Weyl group. Hence $f - \sum_{w \in W} \mathrm{sgn}(w)\, w \cdot e(\mu)$ has strictly smaller weight. The result now follows by induction.

(ii) The equality is routine. Recall that $\{\alpha_1, \ldots, \alpha_\ell\}$ is a base for Φ. It follows from Exercise 3.1(i) and (ii) that

$$\begin{aligned} S_{\alpha_i}\Big(\prod_{\alpha \in \Phi^+} \big(\mathrm{e}(\tfrac{1}{2}\alpha) - \mathrm{e}(-\tfrac{1}{2}\alpha)\big)\Big) &= \frac{-\mathrm{e}(\frac{1}{2}\alpha_i) + \mathrm{e}(-\frac{1}{2}\alpha_i)}{\mathrm{e}(\frac{1}{2}\alpha_i) - \mathrm{e}(-\frac{1}{2}\alpha_i)} \prod_{\alpha \in \Phi^+} \big(\mathrm{e}(\tfrac{1}{2}\alpha) - \mathrm{e}(-\tfrac{1}{2}\alpha)\big) \\ &= - \prod_{\alpha \in \Phi^+} \big(\mathrm{e}(\tfrac{1}{2}\alpha) - \mathrm{e}(-\tfrac{1}{2}\alpha)\big). \end{aligned}$$

Hence the right-hand side is antisymmetric.

(iii) Both sides are anti-symmetric and the coefficients of $\mathrm{e}(\delta)$ agree. The result now follows from (i) since, by Exercise 3.1(iii), δ is the smallest element of Λ_{dom}.

(iv) *Sketch:* it is sufficient to prove that each $a(\lambda)$ is divisible by the product $\prod_{\alpha \in \Phi^+} \big(\mathrm{e}(\frac{1}{2}\alpha) - \mathrm{e}(-\frac{1}{2}\alpha)\big)$. This follows using that $\mathbf{Q}[\frac{1}{2}\Lambda]$ is a UFD.

Exercise 4.4. Let ω be the unique fundamental dominant weight for $\mathsf{sl}_2(\mathbf{C})$, so $\omega \in \langle h \rangle^\star$ is defined by $\omega(h) = 1$.

(i) Use the results of Section 1 to show that V is the irreducible $\mathsf{sl}_2(\mathbf{C})$-module with highest weight $d\omega$ then
$$\chi_V = e(d\omega) + \mathrm{e}((d-2)\omega) + \cdots + \mathrm{e}(-d\omega).$$
(ii) Check that this is consistent with the Weyl Character Formula.

Exercise 4.5. Let ω_1, ω_2 be the fundamental dominant weights for $\mathsf{sl}_3(\mathbf{C})$ found in Exercise 2.8.

(i) Use the Weyl Character Formula to determine the characters of the finite-dimensional irreducible $\mathsf{sl}_3(\mathbf{C})$-module V with highest weight $a\omega_1 + b\omega_2$ where $a, b \in \mathbf{N}_0$.
(ii) Give a necessary and sufficient condition on a and b for V to have a weight space of dimension at least two.

Exercise 5.1. Show that if $f, g, h \in \mathbf{Q}[[\Lambda]]$ then $\{fg, h\} = f\{g, h\} + \{f, h\}g$.

Exercise 5.2. Recall that Q is the denominator in the Weyl Character Formula. Use Exercise 4.3(iii) and Exercise 5.1 to show that

$$2\big\{Q, \mathrm{e}(\nu)\big\} = Q \sum_{\alpha \in \Phi^+} \frac{\mathrm{e}(\alpha) + 1}{\mathrm{e}(\alpha) - 1} (\nu, \alpha)\mathrm{e}(\nu)$$

Solution. By the generalization of Exercise 5.1 to arbitrary products we have

$$\begin{aligned}
2\{Q, \mathrm{e}(\nu)\} &= 2\Big\{\prod_{\alpha\in\Phi^+} \frac{1}{\mathrm{e}(\frac{1}{2}\alpha)-\mathrm{e}(-\frac{1}{2}\alpha)}, \mathrm{e}(\nu)\Big\} \\
&= 2\sum_{\alpha\in\Phi^+} \frac{Q}{\mathrm{e}(\frac{1}{2}\alpha)-\mathrm{e}(-\frac{1}{2}\alpha)}\big\{\mathrm{e}(\tfrac{1}{2}\alpha)-\mathrm{e}(-\tfrac{1}{2}\alpha), \mathrm{e}(\nu)\big\} \\
&= 2\sum_{\alpha\in\Phi^+} \frac{Q}{\mathrm{e}(\frac{1}{2}\alpha)-\mathrm{e}(-\frac{1}{2}\alpha)}\Big((\tfrac{1}{2}\alpha,\nu)\mathrm{e}(\nu+\tfrac{1}{2}\alpha)+(\tfrac{1}{2}\alpha,\nu)\mathrm{e}(\nu-\tfrac{1}{2}\alpha)\Big) \\
&= \sum_{\alpha\in\Phi^+} \frac{Q}{\mathrm{e}(\frac{1}{2}\alpha)-\mathrm{e}(-\frac{1}{2}\alpha)}(\alpha,\nu)\big(\mathrm{e}(\tfrac{1}{2}\alpha)+\mathrm{e}(-\tfrac{1}{2}\alpha)\big)\mathrm{e}(\nu) \\
&= Q\sum_{\alpha\in\Phi^+} \frac{\mathrm{e}(\frac{1}{2}\alpha)+\mathrm{e}(-\frac{1}{2}\alpha)}{\mathrm{e}(\frac{1}{2}\alpha)-\mathrm{e}(-\frac{1}{2}\alpha)}(\nu,\alpha)\mathrm{e}(\nu) \\
&= Q\sum_{\alpha\in\Phi^+} \frac{\mathrm{e}(\alpha)+1}{\mathrm{e}(\alpha)-1}(\nu,\alpha)\mathrm{e}(\nu)
\end{aligned}$$

as required.

Exercise 6.1. Show that the type A root system of sl_n of dimension $n-1$ with $\Phi^+ = \{\varepsilon_i - \varepsilon_j : 1 \le i < j \le n\}$ we have $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ for each i. Using Exercise 3.1(iii) deduce that

$$\delta = \tfrac{1}{2}\sum_{i=1}^{n}(n+1-2i)\varepsilon_i = \sum_{i=1}^{n-1}(n-i)\varepsilon_i.$$

(To obtain the second form, recall that in Λ we have the relation $\varepsilon_1 + \cdots + \varepsilon_n = 0$.) Deduce from the second form that $\delta \in \Lambda$, as expected from Exercise 3.1(iii). Show also that $\delta = \frac{1}{2}\sum_{i=1}^{n-1} i(n-i)(\varepsilon_i - \varepsilon_{i+1})$ and deduce that $\delta \in \Phi^+$ if n is odd and that $\delta \in \frac{1}{2}\Phi^+\backslash\Phi^+$ if n is even.

Exercise 6.2. Recall that $\mathsf{sp}_{2n}(\mathbf{C})$ is defined with respect to the bilinear form J shown in the margin. Show that

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

$$\mathsf{sp}_{2n}(\mathbf{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a^{\mathrm{tr}} \end{pmatrix} : b = -b^{\mathrm{tr}}, c = -c^{\mathrm{tr}} \right\}.$$

Solution. Let $z \in \mathsf{gl}_{2n}(\mathbf{C})$ have block decomposition $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We have

$$\begin{aligned}
z^{\mathrm{tr}}J + Jz &= \begin{pmatrix} a^{\mathrm{tr}} & c^{\mathrm{tr}} \\ b^{\mathrm{tr}} & d^{\mathrm{tr}} \end{pmatrix}\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} + \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \begin{pmatrix} -c^{\mathrm{tr}} & a^{\mathrm{tr}} \\ -d^{\mathrm{tr}} & b^{\mathrm{tr}} \end{pmatrix} + \begin{pmatrix} c & d \\ -a & -b \end{pmatrix}
\end{aligned}$$

hence $z^{\mathrm{tr}}J + Jz = 0$ if and only if $a = -d^{\mathrm{tr}}$, $b^{\mathrm{tr}} = b$ and $c^{\mathrm{tr}} = c$.

Exercise 6.3. Show that in Type C we have $\delta = \sum_{i=1}^{n}(n+1-i)\varepsilon_i$.

Exercise 6.4. Fix the diagonal matrix

$$D = \begin{pmatrix} d_1 & \cdot & \cdot & \cdot \\ \cdot & d_2 & \cdot & \cdot \\ \cdot & \cdot & -d_1 & \cdot \\ \cdot & \cdot & \cdot & -d_2 \end{pmatrix}$$

in the Cartan subalgebra of $\mathsf{sp}_4(\mathbf{C})$.

(i) Show that the following commutator relations hold in $\mathsf{sp}_4(\mathbf{C})$

$$\left[D, \begin{pmatrix} 0 & 1 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & 1 & 0 \end{pmatrix} \right] = (d_1 - d_2) \begin{pmatrix} 0 & 1 & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & 1 & 0 \end{pmatrix}$$

$$\left[D, \begin{pmatrix} \cdot & \cdot & 1 & 0 \\ \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \right] = 2d_1 \begin{pmatrix} \cdot & \cdot & 1 & 0 \\ \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\left[D, \begin{pmatrix} \cdot & \cdot & 0 & 1 \\ \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \right] = (d_1 + d_2) \begin{pmatrix} \cdot & \cdot & 0 & 1 \\ \cdot & \cdot & -1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

and hence show that the root spaces for the positive roots $\varepsilon_1 - \varepsilon_2$, $\varepsilon_1 + \varepsilon_2$, $2\varepsilon_1$, $2\varepsilon_2$ are as claimed just before (6.3). The other root spaces follow by transposing to negate the root.

(ii) Set $\alpha = \varepsilon_1 - \varepsilon_2$ and $\beta = 2\varepsilon_2$. Verify that $h_\alpha = \mathrm{diag}(1, -1, -1, 1)$ an $h_\beta = (0, 1, 0, -1)$, as claimed in (6.4). Deduce that $\langle \alpha, \beta \rangle = \beta(h_\alpha) = -2$ and $\langle \beta, \alpha \rangle = \alpha(h_\beta) = -1$ and hence verify that the Dynkin diagram for the rank 2 type C root system with base α, β is as claimed.

(iii) Verify that $\delta = 2\varepsilon_1 + \varepsilon_2$, reproving a special case of Exercise 6.3.

References

[1] Peter S. Campbell and Anna Stokke, *Hook-content formulae for symplectic and orthogonal tableaux*, Canad. Math. Bull. **55** (2012) 462–473.

[2] Karin Erdmann and Mark J. Wildon, *Introduction to Lie algebras*, Springer Undergraduate Mathematics Series, Springer-Verlag London, Ltd., London, 2006.

[3] Roe Goodman and Nolan R. Wallach, *Symmetry, representations, and invariants*, Graduate Texts in Mathematics, vol. 255, Springer, Dordrecht, 2009.

[4] James E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York-Berlin, 1978, revised Second printing.

[5] Kiyoshi Igusa, *Notes for Math 23a: Lie Algebras, Fall 2011, Section 24b, Weyl Character Formula*. Available from `http://people.brandeis.edu/~igusa/Math223aF11/Notes223a24b.pdf`.

[6] Nathan Jacobson, *Lie algebras*, Dover Publications, Inc., New York, 1979, Republication of the 1962 original.

[7] G. James and A. Kerber, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and its Applications, vol. 16, Addison-Wesley Publishing Co., Reading, Mass., 1981.

[8] Nicholas A. Loehr, *Abacus proofs of Schur function identities*, SIAM J. Discrete Math. **24** (2010), no. 4, 1356–1370.

[9] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999, With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.