# VERTICES OF SPECHT MODULES AND BLOCKS OF THE SYMMETRIC GROUP

MARK WILDON

ABSTRACT. This paper studies the vertices, in the sense defined by J. A. Green, of Specht modules for symmetric groups. The main theorem gives, for each indecomposable non-projective Specht module, a large subgroup contained in one of its vertices. A corollary of this theorem is a new way to determine the defect groups of symmetric groups. The main theorem is also used to find the Green correspondents of a particular family of simple Specht modules; as a corollary, this gives a new proof of the Brauer correspondence for blocks of the symmetric group. The proof of the main theorem uses the Brauer homomorphism on modules, as developed by M. Broué, together with combinatorial arguments using Young tableaux.

## 1. INTRODUCTION

In this paper we apply the methods of local representation theory to the symmetric group. Our object is twofold: firstly to prove Theorem 1.1 below on the vertices of Specht modules, and secondly to use this theorem to give short proofs of two earlier results on the blocks of symmetric groups. Specifically, we determine their defect groups, and how blocks of symmetric groups relate under the Brauer correspondence to blocks of local subgroups of symmetric groups. Our main theorem applies to all Specht modules, but is strongest for partitions of the form $(n-m, m)$: we discuss the special case $(n-2, 2)$ at the end of the paper.

Vertices were first defined in an influential paper of J. A. Green [9]. We recall his definition here. Let $G$ be a finite group and let $F$ be a field of prime characteristic $p$. Let $M$ be an indecomposable $FG$-module. A subgroup $Q$ of $G$ is said to be a *vertex* of $M$ if there is an indecomposable $FQ$-module $N$ such that $V$ is a summand of the induced module $N\uparrow_Q^G$, *and* $Q$ is minimal with this property. By [9, page 435], the vertices of $M$ are $p$-groups, and any two vertices of $M$ are conjugate in $G$. The module $N$ is well-defined up to conjugacy in $N_G(Q)$; it is referred to as the *source* of $V$.

Despite the central role played by vertices in open problems in modular representation theory, such as Alperin's Weight Conjecture [2], little is

known about the vertices of 'naturally occurring' modules, such as Specht modules for symmetric groups. See §3 below for the definition of Specht modules, and other prerequisite results concerning tableaux and blocks of symmetric groups. We recall here that if $\lambda$ is a partition of $n$, then the Specht module $S^\lambda$, defined over a field of zero characteristic, affords the ordinary irreducible character of the symmetric group $S_n$ canonically labelled by $\lambda$. When defined over fields of prime characteristic, Specht modules usually fail to be irreducible. However, by Theorem 3.2, they are usually indecomposable. Our main result is a step towards finding their vertices.

**Theorem 1.1.** *Let $\lambda$ be a partition and let $t$ be a $\lambda$-tableau. Let $H(t)$ be the subgroup of the row-stabilising group of $t$ which permutes, as blocks for its action, the entries of columns of equal length in $t$. If the Specht module $S^\lambda$, defined over a field of prime characteristic $p$, is indecomposable, then it has a vertex containing a Sylow $p$-subgroup of $H(t)$.*

For example, if $\lambda = (8, 4, 1)$ and

$$t = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\\hline 9 & 10 & 11 & 12 \\\cline{1-4} 13 \\\cline{1-1}\end{array}$$

then the row-stabilising group of $t$ is $S_{\{1,2,3,4,5,6,7,8\}} \times S_{\{9,10,11,12\}}$ and $H(t)$ is generated by the permutations

$$(2, 3, 4)(10, 11, 12), (2, 3)(10, 11), (5, 6, 7, 8), (5, 6).$$

Consider as an abstract group, $H(t) \cong S_3 \times S_4$.

Our proof of Theorem 1.1, which is given in §5 below, uses the Brauer homomorphism on modules, as developed by M. Broué in [3]. We briefly state the main results we need from his work in §2. We also use a combinatorial result which refines the Standard Basis Theorem on Specht modules: see Proposition 4.1 below.

We single out the following corollary of Theorem 1.1

**Corollary 1.2.** *If the Specht module $S^\lambda$, defined over a field of characteristic $p$, is indecomposable, then it has a vertex containing a Sylow $p$-subgroup of $S_{\lambda_1 - \lambda_2}$.*                                                                    □

In §6 and §7 we use Corollary 1.2 to give new proofs of two results on the block theory of the symmetric group. We shall suppose in these sections that the reader has some familiarity with block theory: see [1, Chapter 4] for an introduction. We recall here that if $B$ is a $p$-block of the finite group $G$ then,

when thought of as an $F(G \times G)$-module with the action $x(g, g') = g^{-1}xg'$ for $x \in B$ and $g, g' \in G$, $B$ has a vertex of the form

$$\Delta D = \{(g, g) : g \in D\}$$

for some subgroup $D$ of $G$. We say that $D$ is a *defect group* of the block $B$. By [1, §13, Theorem 5], if $M$ is an indecomposable module lying in a block $B$ then $M$ has a vertex contained in a defect group of $B$.

Our results are obtained by considering a particular family of Specht modules. Given $w \in \mathbf{N}$ and a partition $\gamma = (\gamma_1, \ldots, \gamma_k)$ which is a $p$-core (see §3.3 below), let

$$(1) \qquad\qquad \gamma + wp = (\gamma_1 + wp, \gamma_2, \ldots, \gamma_k).$$

We shall say that the partitions $\gamma + wp$ are *initial*. In §6 we use Corollary 1.2 to determine the vertices of Specht modules labelled by initial partitions. This gives a new way to determine the defect groups of blocks of the symmetric group. The ideas in this proof can also be used to give a short proof of Brauer's Height Zero Conjecture for the symmetric group. We explain this in §6.1.

In §7 we find the Green correspondents of Specht modules labelled by initial partitions. This leads to a new way to determine the behaviour of blocks of the symmetric group under the Brauer correspondence. (This was first decided by M. Broué in [4].) By Lemma 7.1, each Specht module $S^{\gamma+wp}$ is simple, so our result is also a first step in finding the Green correspondents of the simple modules of the symmetric groups. For other results on the vertices of particular simple modules, see [7] and [15].

Most of the existing work on the vertices of Specht modules has been on Specht modules labelled by partitions of the form $(n - m, 1^m)$. Their vertices were found by the author in [23, Theorem 2] in the case where the field characteristic does not divide $n$. The remaining case was solved in [18] for fields of characteristic 2; it is an open problem when $m \geq 2$ for fields of odd characteristic.

Theorem 1.1 is at its strongest for Specht modules labelled by two-part partitions. In §8 we use Theorem 1.1 to find the vertices of the Specht modules $S^{(n-2,2)}$ defined over fields of odd characteristic. The harder case of characteristic 2 was recently solved by Danz and Erdmann in [6]. The case of a general two-part partition will be the subject of a later paper.

We end by noting that if $\lambda'$ denotes the conjugate partition to $\lambda$, then by [12, Theorem 8.15], $S^{\lambda'}$ is isomorphic to the dual of $S^\lambda \otimes \mathrm{sgn}$. Hence $S^\lambda$ and

$S^{\lambda'}$ have a vertex in common. In some cases, Theorem 1.1 gives a stronger, or different, result when applied to $S^{\lambda'}$ rather than $S^\lambda$.

## 2. The Brauer homomorphism

Let $G$ be a finite group and let $F$ be a field of prime characteristic $p$. Let $M$ be an $FG$-module. For $Q \leq G$, let $M^Q$ denote the subspace of $M$ consisting of those vectors fixed by every element of $Q$. Given subgroups $R \leq Q \leq G$, we define the *relative trace* map $\mathrm{Tr}_R^Q : M^R \to M^Q$ by

$$\mathrm{Tr}_R^Q(x) = \sum_{i=1}^m x g_i$$

where $g_1, \ldots, g_m$ is a transversal for the right cosets of $R$ in $Q$. The *Brauer quotient* of $M$ with respect to $Q$ is the quotient space

$$M(Q) = M^Q / \sum_{R < Q} \mathrm{Tr}_R^Q V^R.$$

The *Brauer homomorphism* with respect to $Q$ is the quotient map $M^Q \twoheadrightarrow M(Q)$. An easy calculation shows that both $M^Q$ and $\sum_{R < Q} \mathrm{Tr}_R^Q V^R$ are $N_G(Q)$-invariant, and so $M(Q)$ is a module for $F N_G(Q)/Q$.

The next theorem shows how the Brauer homomorphism may be used to garner information about vertices. It is proved in [3, (3.1)].

**Theorem 2.1.** *Let $G$ be a finite group, let $F$ be a field of prime characteristic $p$, and let $M$ be an indecomposable $FG$-module. Let $Q$ be a $p$-subgroup of $G$. If $M(Q) \neq 0$ then $M$ has a vertex containing $Q$.* □

We shall also use the following theorem, which combines results from Theorem 3.2 of [4] and Exercise 27.4 of [22].

**Theorem 2.2.** *Let $G$ be a finite group, let $F$ be a field of prime characteristic $p$, and let $M$ be an indecomposable $FG$-module with trivial source.*

*(i) If $Q$ is a $p$-subgroup of $G$ then $M(Q) \neq 0$ if and only if $Q$ is contained in a vertex of $M$.*

*(ii) If $Q$ is a vertex of $M$ then $M(Q)$ is a projective $F N_G(Q)/Q$-module. Moreover, when regarded as an $F N_G(Q)$-module, $M(Q)$ is the Green correspondent of $M$.* □

Theorem 2.2 cannot be extended to modules which do not have trivial source. For example, if $G$ is cyclic of order 4 and $M$ is the unique indecomposable $\mathbf{F}_2 G$-module of dimension 3, then $M(G) = 0$, even though $M$ has $G$ as its vertex. It is an interesting feature of our proof of Theorem 1.1 that we successfully apply the Brauer homomorphism to modules which are—in most cases—not trivial source.

## 3. Background results on the symmetric group

In this section we collect the prerequisite definitions and results we need from the representation theory of the symmetric group.

3.1. **Tableaux.** Let $\lambda$ be a partition of $n$. A $\lambda$-*tableau* is an assignment of the numbers $\{1, 2, \ldots, n\}$ to the boxes of the Young diagram of $\lambda$, so that each box has a different entry. We say that a $\lambda$-tableau is *row-standard* if its rows are increasing when read from left to right, and *column-standard* if its columns are increasing when read from top to bottom. A tableau that is both row-standard and column-standard is said to be *standard*.

If $u$ is a tableau, then we denote by $\overline{u}$ the row-standard tableau obtained from $u$ by sorting its rows in increasing order. We say that $\overline{u}$ is the *row-straightening* of $u$. For example, if

$$
u = \begin{array}{|c|c|c|c|} \hline 4 & 7 & 6 & 1 \\ \hline 2 & 5 & 3 \\ \cline{1-3} 8 \\ \cline{1-1} \end{array} \quad \text{then} \quad \overline{u} = \begin{array}{|c|c|c|c|} \hline 1 & 4 & 6 & 7 \\ \hline 2 & 3 & 5 \\ \cline{1-3} 8 \\ \cline{1-1} \end{array} \; .
$$

Of the many ways to order the set of standard tableaux, the most fundamental is the dominance order. It will be useful to define this order on the larger set of row-standard tableaux. First though, we must define the dominance order on compositions: if $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \ldots, \mu_m)$ are compositions of $n$, then we say that $\lambda$ *dominates* $\mu$, and write $\lambda \trianglerighteq \mu$, if

$$
\lambda_1 + \cdots + \lambda_r \geq \mu_1 + \cdots + \mu_r
$$

for all $r \in \mathbf{N}$. (If $r$ exceeds the number of parts of $\lambda$ or $\mu$, then take the corresponding part to be 0.) If $t$ is a row-standard tableau, then we denote by $\mathrm{sh}(t^{\leq j})$ the composition recording the number of entries $\leq j$ in each row of $t$. For example if $t = \overline{u}$ where $u$ is as above, then $\mathrm{sh}(t^{\leq 8}) = (4, 3, 1)$ and $\mathrm{sh}(t^{\leq 5}) = (2, 3, 0)$. Finally, if $\lambda$ is a partition of $n$ and $s$ and $t$ are row-standard $\lambda$-tableaux, then we say that $s$ *dominates* $t$ if

$$
\mathrm{sh}\big(s^{\leq j}\big) \trianglerighteq \mathrm{sh}\big(t^{\leq j}\big)
$$

for all $j$ with $1 \leq j \leq n$. Following the usual convention, we shall reuse the $\trianglerighteq$ symbol for the dominance order on row-standard tableaux.

3.2. **Specht modules.** We briefly recall the definition of the Specht module $S^\lambda$ as a submodule of the Young permutation module $M^\lambda$. The reader is referred to [12] for examples and further details.

Let $\lambda$ be a partition of $n$. Given a $\lambda$-tableau $t$, we obtain the associated *tabloid* $\mathbf{t}$ by disregarding the order of the elements within the rows of $t$. For

example, if

$$t = \begin{array}{|c|c|c|c|}\hline 4 & 7 & 6 & 1 \\\hline 2 & 5 & 3 \\\cline{1-3} 8 \\\cline{1-1}\end{array} \quad \text{then} \quad \mathbf{t} = \begin{array}{cccc} 4 & 7 & 6 & 1 \\ \hline 2 & 5 & 3 \\ \hline 8 \\ \hline \end{array} = \begin{array}{cccc} 1 & 4 & 6 & 7 \\ \hline 2 & 3 & 5 \\ \hline 8 \\ \hline \end{array} = \ldots \quad \text{etc.}$$

The natural action of $S_n$ on the set of $\lambda$-tableaux gives rise to a well-defined action of $S_n$ on the set of $\lambda$-tabloids. We denote the associated permutation representation of $S_n$ by $M^\lambda$; it is the *Young permutation module* corresponding to $\lambda$. If we need to emphasise that the ground ring is $R$, then we shall write $M_R^\lambda$. For example, $M_{\mathbf{Z}}^{(n-1,1)}$ affords the natural integral representation of $S_n$ as $n \times n$ permutation matrices.

Given a $\lambda$-tableau $t$, we let $C(t)$ be the subgroup of $S_n$ consisting of those elements which fix setwise the columns of $t$. The *polytabloid* corresponding to $t$ is the element $e_t$ of $M^\lambda$ defined by

$$e_t = \sum_{g \in C(t)} \mathbf{t}g \, \mathrm{sgn}(g).$$

The *Specht module* $S^\lambda$ is defined to be the submodule of $M^\lambda$ spanned by the $\lambda$-polytabloids. Again, we write $S_R^\lambda$ if we need to emphasise the ground ring. An easy calculation shows that if $h \in S_n$ then $(e_t)h = e_{th}$, and so $S^\lambda$ is cyclic, generated by any single polytabloid. Moreover, if $h \in C(t)$ then

$$(2) \qquad\qquad\qquad (e_t)h = \mathrm{sgn}(h)e_t.$$

It follows easily from (2) that $S^\lambda$ is linearly spanned by the polytabloids $e_t$ for which $t$ is column-standard. More is true: in the statement of the following theorem, if $t$ is a standard tableau, then we say that $e_t$ is a *standard polytabloid*.

**Theorem 3.1** (Standard Basis Theorem)**.** *The standard $\lambda$-polytabloids form a $\mathbf{Z}$-basis for the integral Specht module $S_{\mathbf{Z}}^\lambda$.* $\qquad\qquad\square$

A short proof of the Standard Basis Theorem, attributed to J. A. Green, was given in [20, §3]. It is presented with some simplifications in [12, Chapter 8]. The corresponding result for Specht modules defined over fields is an immediate corollary.

The next theorem gives two sufficient conditions for a Specht module to be indecomposable.

**Theorem 3.2.** *Let $F$ be a field of prime characteristic $p$ and let $\lambda$ be a partition of $n$. If $p > 2$, or if $p = 2$ and the parts of $\lambda$ are distinct, then $S_F^\lambda$ is indecomposable.*

*Proof.* When $p > 2$ it is well known that $\mathrm{End}_{FS_n}(S^\lambda) \cong F$. (This result has a particularly short proof using the alternative definition of polytabloids as polynomials: see [20, Theorem 4.1] or [21]. For stronger results in this direction, see [12, Chapter 13].) When the second condition holds, it follows from Theorems 4.9 and 11.1 in [12] that the top of $S^\lambda$ is simple. Hence in both cases $S_F^\lambda$ is indecomposable. $\square$

When $p = 2$, it is possible for Specht modules to be decomposable. G. D. James gave the first example in [11] where he showed that $S_{\mathbf{F}_2}^{(5,1,1)}$ is decomposable. In [17], G. M. Murphy showed that this was the first of infinitely many examples by giving a necessary and sufficient condition for the Specht module $S_{\mathbf{F}_2}^{(2m+1-r,1^r)}$ to be decomposable. In Proposition 3.3.2 of [24], the author used Theorem 2 in [23] on the vertices of Specht modules labelled by hook partitions to give a shorter proof of Murphy's result. It is an open question whether there are any decomposable Specht modules other than those found by Murphy.

### 3.3. Blocks of symmetric groups.

The blocks of symmetric groups are described by a theorem which seems destined to remain forever known as Nakayama's Conjecture. In order to state it we must first recall some definitions.

Let $\lambda$ be a partition. A *p-hook* in $\lambda$ is a connected part of the rim of the Young diagram of $\lambda$ consisting of exactly $p$ boxes, whose removal leaves the diagram of a partition. By repeatedly stripping off $p$-hooks from $\lambda$ we obtain the *p-core* of $\lambda$; the number of hooks we remove is the *weight* of $\lambda$. For an example, see Figure 1. Often it is best to perform these operations using James' *abacus*: for a description of how to use this piece of apparatus, see [10, pages 76–78]. For instance, it is easy to prove *via* the abacus that the $p$-core of a partition is well defined, something which is otherwise not at all obvious.
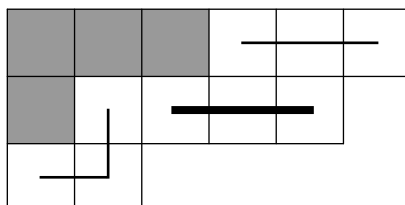


**Figure 1.** The 3-core of $(6, 5, 2)$ is $(3, 1)$. The thick line indicates a 3-hook in $(6, 5, 2)$; the other two lines show 3-hooks of partitions obtained *en route* to the 3-core.

**Theorem 3.3** (Nakayama's Conjecture)**.** *Let $p$ be a prime. The $p$-blocks of the symmetric group $S_n$ are labelled by pairs $(\gamma, w)$, where $\gamma$ is a $p$-core and $w \in \mathbf{N}_0$ is the associated weight, such that $|\gamma| + wp = n$. Thus $S^\lambda$ lies in the block labelled by $(\gamma, w)$ if and only if $\lambda$ has $p$-core $\gamma$ and weight $w$.* □

Many proofs of Nakayama's Conjecture are now known. A particularly elegant proof was given by Broué in [4] using Brauer pairs. Proposition 2.12 in [4] states the following result describing the defect groups of blocks of symmetric groups. We shall use vertices to give an alternative proof in §6 below.

**Theorem 3.4.** *Let $p$ be a prime. If $B$ is a $p$-block of $S_n$ of weight $w$ then the defect group of $B$ is a Sylow $p$-subgroup of $S_{wp}$.* □

## 4. A STRAIGHTENING RULE

The object of this section is to prove the refinement of the Standard Basis Theorem (Theorem 3.1) stated in Proposition 4.1 below. It seems slightly surprising that this proposition is not already known; since it appears to be the sharpest possible result in its direction, the author believes that it is well worth putting it on record.

**Proposition 4.1.** *Let $\lambda$ be a partition. If $u$ is a column-standard $\lambda$-tableau then its row-straightening $\overline{u}$ is a standard tableau. Moreover, in $S_{\mathbf{Z}}^\lambda$,*

$$e_u = e_{\overline{u}} + x$$

*where $x$ is an integral linear combination of standard polytabloids $e_v$ for tableaux $v$ such that $\overline{u} \rhd v$.*

*Proof.* By construction, $\overline{u}$ is row-standard, so to prove the first part of the proposition, it suffices to show that $\overline{u}$ is also column-standard. Let $\overline{u}_{i,j}$ denote the entry of $\overline{u}$ in row $i$ and column $j$. Suppose, for a contradiction, that $\overline{u}_{i+1,j} < \overline{u}_{i,j}$. Let

$$A = \{\overline{u}_{i+1,1}, \ldots, \overline{u}_{i+1,j}\},$$
$$B = \{\overline{u}_{i,j}, \ldots, \overline{u}_{i,\lambda_i}\}.$$

The entries of $A$ lie in row $i + 1$ of $u$ and the entries of $B$ lie in row $i$ of $u$. Since $|A| + |B| = \lambda_i + 1$, there exists $a \in A$ and $b \in B$ such that $a$ and $b$ appear in the same column of $u$. But then

$$a \leq \overline{u}_{i+1,j} < \overline{u}_{i,j} \leq b$$

which contradicts the hypothesis that $u$ is column-standard.

To prove the second part of the proposition, it will be useful to define the dominance order on tabloids: given $\lambda$-tabloids $\mathbf{s}$ and $\mathbf{t}$ corresponding to the tableaux $s$ and $t$ respectively, we set $\mathbf{s} \trianglerighteq \mathbf{t}$ if and only if $\overline{s} \trianglerighteq \overline{t}$. We shall need Lemma 8.3 in [12], which states that if $t$ is a column-standard tableau, then

$$(3) \qquad\qquad e_t = \mathbf{t} + y$$

where $y$ is an integral linear combination of tabloids $\mathbf{v}$ such that $\mathbf{t} \triangleright \mathbf{v}$.

By the Standard Basis Theorem (Theorem 3.1), there exist integers $\alpha_v \in \mathbf{Z}$ such that

$$e_u = \sum \alpha_v e_v$$

where the sum is over all standard tableaux $v$. Let $w$ be maximal in the dominance order such that $\alpha_w \neq 0$. By (3), applied with $t = w$, the tabloid $\mathbf{w}$ appears with coefficient $\alpha_w$ in $e_u$. Another application of (3), this time with $t = u$, shows that $\mathbf{u} \trianglerighteq \mathbf{w}$. Hence

$$(4) \qquad\qquad e_u = \alpha_{\overline{u}} e_{\overline{u}} + x$$

where $x$ is an integral linear combination of standard polytabloids $e_v$ for tableaux $v$ such that $\overline{u} \triangleright v$.

By (3), the tabloid $\mathbf{u}$ appears in $e_u$ with coefficient 1. Again by (3), this tabloid cannot appear in the summand $x$ in (4). It follows that $\alpha_{\overline{u}} = 1$, as required. $\qquad\square$

In addition to our main theorem, Proposition 4.1 may also be used to give a very short proof of the key Lemma 2.1 in G. E. Murphy's paper [16], which states that if $t$ is a standard tableau and $a$ and $b$ are entries of $t$ such that $a < b$ and $a$ appears to the right of $b$ in $t$, then $e_t(ab)$ is an integral linear combination of polytabloids $e_s$ for standard tableaux $s$ such that $t \triangleright s$. Indeed, since $t \triangleright \overline{t(ab)}$, Proposition 4.1 gives the stronger result that $\overline{t(ab)} \trianglerighteq s$.

It is interesting to note that the special case of the first part of the proposition in which $\lambda$ is a rectangular partition is given—as an exercise in the Pigeonhole Principle—in §10.7 of [5].

## 5. Proof of the main theorem

We are now ready to prove Theorem 1.1. We shall show that if $P$ is a Sylow $p$-subgroup of $H(t)$ then $S^\lambda(P) \neq 0$. The first step is given by the following lemma, which implies that $e_t \in (S^\lambda)^P$.

**Lemma 5.1.** *Let $t$ be a $\lambda$-tableau. The polytabloid $e_t$ is fixed by every permutation in $H(t)$.*

*Proof.* Let $h \in H(t)$. By definition, $h$ permutes the columns of $t$ as blocks for its action, so $C(t)^h = C(t)$. Moreover, $\mathbf{t}h = \mathbf{t}$. Hence

$$e_t h = \sum_{g \in C(t)} \mathbf{t}gh\,\mathrm{sgn}(g) = \sum_{x \in C(t)^h} \mathbf{t}hx\,\mathrm{sgn}(x) = \sum_{x \in C(t)} \mathbf{t}x\,\mathrm{sgn}(x) = e_t,$$

as required.                                                                    $\square$

To proceed further it will be useful to make a specific choice for the $\lambda$-tableau $t$. Since the subgroups $H(t)$ for different tableau $t$ are conjugate, this incurs no loss in generality. We shall take $t$ to be the greatest $\lambda$-tableau in the dominance order; thus if $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $1 \le j \le k$, then the entries of $t$ in its $j$th row are

$$R_j = \{\lambda_1 + \cdots + \lambda_{j-1} + 1, \ldots, \lambda_1 + \cdots + \lambda_j\}.$$

For example, if $\lambda = (8, 4, 1)$ then $t$ is the tableau shown after the statement of Theorem 1.1.

**Lemma 5.2.** *Let $t$ be the greatest $\lambda$-tableau in the dominance order and let $Q$ be a p-subgroup of $H(t)$. The polytabloid $e_t$ is not contained in the kernel of the Brauer homomorphism from $(S^\lambda)^Q$ to $S^\lambda(Q)$.*

*Proof.* Let

$$U = \sum_{R < Q} \mathrm{Tr}_R^Q (S^\lambda)^R$$

be the kernel of the Brauer homomorphism with respect to $Q$. In the sum defining $U$ it suffices to take only those subgroups $R$ which are maximal subgroups of $Q$, for if $R' < R < Q$ then

$$\mathrm{Tr}_{R'}^Q (S^\lambda)^{R'} = \mathrm{Tr}_R^Q \Big( \mathrm{Tr}_{R'}^R (S^\lambda)^{R'} \Big) \subseteq \mathrm{Tr}_R^P (S^\lambda)^R.$$

Hence, if $V$ is the subspace of $S^\lambda$ defined by

$$V = \big\langle e_s + e_s g + \cdots + e_s g^{p-1} : s \text{ a standard } \lambda\text{-tableau}, g \in Q \big\rangle,$$

then $U$ is contained in $V$. We shall use Proposition 4.1 to show that $e_t \notin V$.

Suppose that there is a standard $\lambda$-tableau $s$, and a permutation $g \in Q$, such that $e_t$ appears with a non-zero coefficient in the expression of

(5)                                 $$e_s + e_s g + \cdots + e_s g^{p-1}$$

as an linear combination of standard polytabloids. Choose $i$ such that $e_t$ appears with a non-zero coefficient in the expression of $e_s g^i$.

Let $u$ be the column-standard tableau whose columns agree setwise with $sg^i$; by (2), $e_u = \pm e_{sg^i}$. It therefore follows from Proposition 4.1 that

$$e_{sg^i} = \pm e_{\overline{u}} + x$$

where $x$ is a linear combination of polytabloids $e_v$ for standard tableaux $v$ such that $\overline{u} \rhd v$. Since $t$ is the greatest tableau in the dominance order, and the standard polytabloids are linearly independent, we must have $t = \overline{u}$.

If two elements lying in the same row of $t$ appear in the same column of $sg^i$, then these elements appear in different rows of $\overline{u}$. Hence $t \neq \overline{u}$, a contradiction. We may therefore assume that, for each $j$, the elements of $R_j$ appear in different columns of $sg^i$. Since $g$ permutes the elements of each set $R_j$, it follows that for each $j$, the elements of $R_j$ appears in different columns of $s$. Since $s$ is standard, row $j$ of $s$ must consist exactly of the elements of $R_j$; that is, $s = t$. But, by Lemma 5.1, $e_t g = e_t$, and so

$$e_s + e_s g + \ldots + e_s g^{p-1} = pe_s = 0.$$

This contradicts our assumption that $e_t$ has a non-zero coefficient in (5). Therefore $e_t \notin V$, as required.                                              $\square$

Lemmas 5.1 and 5.2 imply that $S^\lambda(P) \neq 0$. Theorem 1.1 now follows from Theorem 2.1. The full strength of Lemma 5.2 will be used in §7 below.

## 6. Defect groups of blocks of the symmetric group

We now apply Corollary 1.2 to the initial partitions defined in (1) to give a new proof of Theorem 3.4 on the defect groups of the symmetric group.

Throughout this section, we denote by $[m]_p$ the highest power of $p$ dividing the natural number $m$. We shall need the following general result from block theory, which connects Brauer's original definition of the defect of a block with our definition *via* vertices. For a proof, see [8, Theorem 61.8].

**Theorem 6.1.** *Let $B$ be a $p$-block of a finite group $G$. Let $p^a$ be the highest power of $p$ dividing $|G|$. Suppose that the defect groups of $B$ have order $p^d$. If $\chi$ is an irreducible character lying in $B$, then $p^{a-d}$ divides $\chi(1)$. Moreover, there is an irreducible character $\chi$ lying in $B$ such that $[\chi(1)]_p = p^{a-d}$.*    $\square$

We shall also need a companion result, which has an entirely combinatorial proof.

**Lemma 6.2.** *Let $\gamma$ be a $p$-core and let $w \in \mathbf{N}$. If $[n!]_p = p^a$ and $[(wp)!]_p = p^b$ then*

$$[\dim S^{\gamma + wp}]_p = p^{a-b}.$$

*Moreover, if $\mu$ is any other partition with $p$-core $\gamma$ and weight $w$ then*

$$[\dim S^\mu]_p \geq p^{a-b}.$$

*Proof.* We shall use the $p$-quotient of a partition, as defined in [10, 2.7.29]. By [10, Theorem 2.7.37], if $\lambda$ is a partition with $p$-quotient $(\lambda(0), \ldots, \lambda(p-1))$ then there is a bijection between hooks in $\lambda$ of length divisible by $p$ and the hooks of the $\lambda(i)$; a hook of length $rp$ in $\mu$ corresponds to a hook of length $r$ in one of the $\mu(i)$. The reader may care to verify this for the partition $(6, 5, 2)$ shown in Figure 1, which has $((2), \varnothing, (1))$ as one of its 3-quotients. By definition of the $p$-quotient, $|\lambda(0)| + \cdots + |\lambda(p-1)|$ is the $p$-weight of $\lambda$.

The partition $\gamma + wp$ has $((w), \varnothing, \ldots, \varnothing)$ as a $p$-quotient, and so its hooks of length divisible by $p$ have lengths $p$, $2p$, $\ldots$, $wp$. (This can also be seen directly from its partition diagram.) Hence the highest power of $p$ dividing the product of the hook-lengths of $\gamma + wp$ is $[(wp)!]_p = p^b$. The first part of the lemma now follows from the Hook Formula for the dimension of Specht modules (see [10, Theorem 2.3.21]).

The second part of the lemma will follow from the Hook Formula if we can show that the highest power of $p$ dividing the product of the hook-lengths of $\mu$ is at most $p^b$. Let $(\mu(0), \ldots, \mu(p-1))$ be a $p$-quotient of $\mu$. Suppose that $\mu(i)$ is a partition of $c_i$. Writing $h_\alpha$ for the hook-length of a node $\alpha$ of a partition, we have

$$\prod_{\alpha \in \mu} [h_\alpha]_p = p^w \prod_{i=0}^{p-1} \prod_{\alpha \in \mu(i)} [h_\alpha]_p = p^w \prod_{i=0}^{p-1} \left[ \frac{c_i!}{\dim S^{\mu(i)}} \right]_p.$$

Rearranging and substituting $[w!]_p$ for $p^{b-w}$ we get

$$(6) \qquad p^b \Big/ \prod_{\alpha \in \mu} [h_\alpha]_p = \left[ \binom{w}{c_0, c_1, \ldots, c_{p-1}} \right]_p \prod_{i=0}^{p-1} \left[ \dim S^{\mu(i)} \right]_p.$$

Clearly the right-hand side of (6) is integral, as we required. $\square$

Let $\gamma$ be a $p$-core, let $w \in \mathbf{N}$ and let $n = |\gamma| + wp$. Let $B$ be the $p$-block of $S_n$ with $p$-core $\gamma$ and weight $w$. Applying Corollary 1.2 to the initial partition $\gamma + wp$ defined in (1), we see that there is a vertex of $S^{\gamma+wp}$ containing a Sylow $p$-subgroup of $S_{wp}$. Hence $B$ has a defect group $D$ that contains a Sylow $p$-subgroup of $S_{wp}$. To complete the proof of Theorem 3.4 we must show that $D$ is no larger.

When defined over fields of characteristic zero, Specht modules afford the irreducible characters of $S_n$. It therefore follows from Theorem 6.1 that if $p^d$ is the order of the defect group $D$ and $\mu$ is a partition with $p$-core $\gamma$ and weight $w$, then

$$(7) \qquad\qquad\qquad [\dim S^\mu]_p \geq p^{a-d}$$

where $p^a = [n!]_p$. Moreover, equality holds in (7) for at least one such partition $\mu$. But by Lemma 6.2, if $p^b = [(wp)!]_p$ then

$$[\dim S^\mu]_p \geq p^{a-b}$$

with equality when $\mu = \gamma + wp$. Thus the minimum in (7) occurs when $\mu = \gamma + wp$, and hence $p^d = p^b$. Therefore $D$ has the same order as a Sylow subgroup of $S_{wp}$. This completes the proof of Theorem 3.4.

6.1. **On Brauer's Height Zero Conjecture.** Let $G$ be a finite group and let $\chi$ be an ordinary character of $G$ lying in a $p$-block with defect group of order $p^d$. Let $p^a$ be the highest power of $p$ dividing $|G|$. If

$$[\chi(1)]_p = p^{a-d+h}$$

then we say that $h$ is the *height* of $\chi$. (It follows from Theorem 6.1 that $h \in \mathbf{N}_0$.) R. Brauer made the following conjecture on character heights.

**Conjecture 6.3** (Brauer's Height Zero Conjecture). *Every ordinary irreducible character in a block $B$ of a finite group has height zero if and only if $B$ has an abelian defect group.*

Proposition 3.9 in Olsson's paper [19] on character heights and the McKay Conjecture gives a proof of Brauer's Height Zero Conjecture for symmetric groups. It is worth noting that equation (6) can be used to give a short alternative proof. If $\mu$ is a partition with $p$-quotient $(\mu(0), \ldots, \mu(p-1))$, where $\mu(i)$ is a partition of $c_i$, then by (6),

$$p^h = \left[ \binom{w}{c_0, c_1, \ldots, c_{p-1}} \right]_p \prod_{i=0}^{p-1} \left[ \dim S^{\mu(i)} \right]_p$$

where $h$ is the height of the ordinary character of $S^\mu$.

If $w < p$ then $c_i < p$ for each $i$, and so each $S^{\mu(i)}$ has dimension coprime to $p$. It follows that in blocks of the symmetric group of abelian defect, every ordinary irreducible character has height 0.

If $w \geq p$ then we may choose the $c_i$ so that $c_0 + c_1 + \cdots + c_{p-1} = w$ and the multinomial coefficient is divisible by $p$, and then set $\mu(i) = (c_i)$. Hence in a block of non-abelian defect, there is an ordinary irreducible character of non-zero height.

## 7. The Brauer correspondence for the symmetric group

We now use the Brauer homomorphism to determine the Green correspondents of Specht modules labelled by initial partitions. As a corollary of this result (see Corollary 7.5), we get a complete description of how blocks

of symmetric groups relate, under the Brauer correspondence, to blocks of their local subgroups.

Throughout this section, let $F$ be a field of characteristic $p$, let $\gamma$ be a $p$-core and let $w \in \mathbf{N}$. Let $m = |\gamma|$ and let $n = m + wp$. The following lemma gives some of the convenient properties of the Specht modules $S^{\gamma+wp}$ and $S^\gamma$.

**Lemma 7.1.** *The Specht module $S^{\gamma+wp}$ is a simple $FS_n$-module with trivial source. The Specht module $S^\gamma$ is a simple projective $FS_m$-module.*

*Proof.* Because $\gamma + wp$ is the greatest partition in the dominance order labelling a Specht module in the block with core $\gamma$ and weight $w$, it follows from [12, Corollary 12.2] that $S^{\gamma+wp}$ is simple. Similarly, it follows from [12, Theorem 12.1] that $S^{\gamma+wp}$ is the only module in its block that appears as a composition factor of $M^{\gamma+wp}$. Hence $S^{\gamma+wp}$ is a direct summand of $M^{\gamma+wp}$, and so has trivial source. For the second part, observe that since $\gamma$ is a $p$-core, $S^\gamma$ is an indecomposable module lying in a block of $S_m$ of defect zero. Hence $S^\gamma$ is simple and projective.                □

Our next proposition gives some useful information about the Brauer quotients of Specht modules labelled by initial partitions. In it, we say that a subgroup $G$ of $S_n$ has *support of size $k$* if the number of elements of $\{1, 2, \ldots, n\}$ that are moved by some permutation in $G$ is $k$.

**Proposition 7.2.** *If $Q$ is a $p$-subgroup of $S_{wp}$ with support of size $rp$ then*

$$N_{S_n}(Q)/Q \cong S_{n-rp} \times N_{S_{rp}}(Q)/Q$$

*and $S^{\gamma+wp}(Q)$ contains a $FN_{S_n}(Q)/Q$-submodule isomorphic, under this identification, to*

$$S^{\gamma+(w-r)p} \otimes F.$$

*If $P$ is a Sylow $p$-subgroup of $S_{wp}$ then $S^{\gamma+wp}(P) \cong S^\gamma \otimes F$.*

*Proof.* As in §5, let $t$ be the greatest tableau of shape $\gamma+wp$ in the dominance order on tableaux. By replacing $Q$ with one of its conjugates if necessary, we can assume that $Q$ is contained in the symmetric group on the set

$$Y = \{\gamma_1 + (w-r)p + 1, \ldots, \gamma_1 + wp\}.$$

Let $X = \{1, 2, \ldots, n\}\backslash Y$. Clearly we have $N_{S_n}(Q) = S_X \times N_{S_Y}(Q)$, which implies the first assertion in the theorem.

Let $U$ be the kernel of the Brauer homomorphism from $(S^{\gamma+wp})^Q$ to $S^{\gamma+wp}(Q)$. Since $Q$ is contained in the subgroup $H(t)$ defined in Theorem 1.1, it follows from Lemma 5.2 that $e_t + U$ is a non-zero element of

$S^{\gamma+wp}(Q)$. Let $W$ be the submodule of $S^{\gamma+wp}(Q)$ generated by $e_t + U$. By Lemma 5.1, $e_t$ is fixed by every permutation in $S_Y$, so $N_{S_Y}(Q)/Q$ acts trivially on $W$.

Let $s$ be the greatest tableau of shape $\gamma+(w-r)p$ in the dominance order. Restricting to the action of $S_X \cong S_{n-rp}$, we see that there is a surjective map of $FS_{n-rp}$-modules,

$$S^{\gamma+(w-r)p} \to W$$

defined by extending $e_s \mapsto e_t + U$. This map is non-zero since $e_t \notin U$. Moreover, by Lemma 7.1, $S^{\gamma+(w-r)p}$ is a simple $FS_{n-rp}$-module. Hence

$$W \cong S^{\gamma+(w-r)p} \otimes F$$

as a module for $FN_{S_n}(Q)/Q$.

It only remains to show that if $P$ is a Sylow $p$-subgroup of $S_Y \cong S_{wp}$, then $S^{\gamma+wp}(P) \cong S^\gamma \otimes F$. By the previous paragraph, $S^{\gamma+wp}(P)$ has a submodule $W$ isomorphic to $S^\gamma \otimes F$. By Lemma 7.1, $S^\gamma$ is a projective $FS_m$-module, and since $N_{S_Y}(P)/P$ has order coprime to $p$, the trivial $FN_{S_Y}(P)/P$-module is projective. Hence $W$ is projective as a module for $F(S_X \times N_{S_Y}(P))$, and so it splits off as a direct summand of $S^{\gamma+wp}(P)$. Since $S^{\gamma+wp}$ has trivial source, it follows from Theorem 2.2 that $S^{\gamma+wp}(P)$ is indecomposable (and projective) as an $FN_{S_n}(P)/P$-module. Therefore $S^{\gamma+wp}(P) = W$. The result follows. $\qquad\square$

By Theorem 2.2(ii), the Green correspondent of a trivial source module is equal to its Brauer quotient. Proposition 7.2 therefore implies Theorem 7.3 below, which gives a complete description of the local properties of Specht modules labelled by initial partitions. Since $S^{\gamma+wp}$ agrees with the Young module $Y^{\gamma+wp}$ (see [13] or [14, Definition 4.6.1]), this theorem also follows, after a little work, from results on the vertices of Young modules—see for example [14, Theorem 4.6.3]. It is worth noting that the proof given here is independent of the theory of Young modules.

**Theorem 7.3.** *Let $P$ be a Sylow $p$-subgroup of $S_{wp}$ and let $\gamma$ be a $p$-core. The Specht module $S^{\gamma+wp}$ has $P$ as one of its vertices, and its source is the trivial $FP$-module. The Green correspondent of $S^{\gamma+wp}$ is the $FN_{S_n}(P)$-module $S^\gamma \otimes F$.* $\qquad\square$

We end by using Proposition 7.2 to describe how blocks of symmetric groups behave under the Brauer correspondence. (For the original proof by M. Broué, see [4].) Our definition of the Brauer correspondence is taken from Alperin [1, §14]; thus if $H$ is a subgroup of a finite group $G$ and $b$ is a block of $H$, then we say that $b$ *corresponds to the block $B$ of $G$*, and

write $b^G = B$, if $b$, considered as a $F(H \times H)$-module, is a summand of the restriction of $B$ to $H \times H$, and $B$ is the unique block of $G$ with this property.

We shall need the following lemma, which generalises a well-known result about the Green correspondence (see, for example [1, §14, Corollary 4]) to Brauer quotients.

**Lemma 7.4.** *Let $G$ be a finite group and let $M$ be an indecomposable $FG$-module with vertex $P$ and trivial source. Let $Q$ be a subgroup of $P$. Suppose that $M$ lies in the block $B$ of $G$. If $M(Q)$, considered as an $FN_G(Q)$-module, has a summand in the block $b$ of $N_G(Q)$, then $b^G$ is defined and $b^G = B$.*

*Proof.* By [22, Exercise 27.4], when considered as an $FN_G(Q)$-module, $M(Q)$ has a summand whose vertex contains $Q$. Hence there is some defect group $D$ of $b$ which contains $Q$. Therefore $N_G(Q) \supseteq C_G(Q) \supseteq C_G(D)$, and so by part 3 of Lemma 1 on page 101 of [1], $b^G$ is defined.

Again by [22, Exercise 27.4], $M(Q)$ is a direct summand of $M{\downarrow}_{N_G(Q)}$. Hence $M(Q)$ is not killed by $B{\downarrow}_{N_G(Q) \times N_G(Q)}$, and so

$$B{\downarrow}_{N_G(Q) \times N_G(Q)} \, b \neq 0.$$

Hence $b^G = B$, as required. $\qquad\square$

**Corollary 7.5.** *Let $Q$ be a $p$-subgroup of $S_{wp}$ with support of size $rp$. In the Brauer correspondence between blocks of $S_n$ and blocks of $N_{S_n}(Q) \cong S_{n-rp} \times N_{S_{rp}}(Q)$, the $p$-block of $S_n$ with core $\gamma$ and weight $w$ corresponds to*

$$b \otimes b_0(N_{S_{rp}}(Q))$$

*where $b$ is the $p$-block of $S_{n-rp}$ with core $\gamma$ and weight $w-r$, and $b_0(N_{S_{rp}}(Q))$ is the principal block of $N_{S_{rp}}(Q)$.*

*Proof.* It follows from Proposition 7.2 that $S^{\gamma+wp}(Q)$ has a summand in the block $b \otimes b_0(N_{S_{wp}}(Q))$. Now apply Lemma 7.4. $\qquad\square$

In his earlier proof of Corollary 7.5, Broué notes that the group $N_{S_{rp}}(Q)$ has a unique $p$-block (see [4, Lemma 2.6]). The correspondence described in Corollary 7.5 is therefore bijective.

## 8. The vertices of $S^{(n-2,2)}$

We end by using Theorem 1.1 to find the vertices of $S^{(n-2,2)}$ over fields of odd characteristic. We shall need Corollary 1 of [9], which states that if $G$ is a finite group, $F$ is a field of characteristic $p$, and $M$ is an indecomposable $FG$-module of dimensional coprime to $p$, then $M$ has a Sylow $p$-subgroup of $G$ as one of its vertices. Our result is as follows.

**Theorem 8.1.** *Let $n \geq 4$. When defined over a field of odd characteristic $p$, the Specht module $S^{(n-2,2)}$ is indecomposable, and its vertex is equal to the defect group of the $p$-block in which it lies.*

*Proof.* That $S^{(n-2,2)}$ is indecomposable follows from Theorem 3.2. The dimension of $S^{(n-2,2)}$ is $n(n-3)/2$. Hence if neither $n$ nor $n-3$ is divisible by $p$, then, by the result just mentioned, $S^{(n-2,2)}$ has a Sylow $p$-subgroup of $S_n$ as its vertex. This is also a defect group of its $p$-block.

If $p$ divides $n$ and $p > 3$ then $(n-2,2)$ has $p$-core $(p-2,2)$ and weight $n/p-1$. Hence the defect group of its block is a Sylow $p$-subgroup of $S_{n-p}$. It follows from Theorem 1.1 that $S^{(n-2,2)}$ has a vertex containing a Sylow $p$-subgroup of $S_{n-p}$. Therefore the vertex of $S^{(n-2,2)}$ agrees with its defect group. If $p = 3$ then $(n-2,2)$ lies in the block with $p$-core $(4,2)$ and weight $(n-6)/3$. Theorem 1.1 implies that there is a vertex containing a Sylow 3-subgroup of $S_{n-6}$, as required.

The only remaining case is when $p$ divides $n-3$ and $p > 3$. In this case $(n-2,2)$ has $p$-core $(p+1,2)$ and weight $(n-3)/p-1$, and the result follows from Theorem 1.1 as before. $\qquad\square$

## Acknowledgements

## References

[1] J. L. Alperin, *Local representation theory*, vol. 11 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 1986.

[2] J. L. Alperin, Weights for finite groups. In *The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986)*, *Proc. Sympos. Pure Math. 47*. Amer. Math. Soc. (1987), 369–379.

[3] M. Broué, On Scott modules and $p$-permutation modules: an approach through the Brauer homomorphism. *Proc. Amer. Math. Soc. 93* (1985), 401–408.

[4] M. Broué, Les $l$-blocs des groupes $GL(n,q)$ et $U(n,q^2)$ et leurs structures locales. *Astérisque 133–134* (1986), 159–188.

[5] P. J. Cameron, *Combinatorics: Topics, Techniques, Algorithms*. Cambridge University Press, 1994.

[6] S. Danz, K. Erdmann, The vertices of a class of Specht modules and simple modules for symmetric groups in characteristic 2. *Preprint* (May 2009). Available from `http://www.minet.uni-jena.de/algebra/preprints/Danz/twopartchar2.pdf`.

[7] S. Danz, B. Külshammer, R. Zimmermann, On vertices of simple modules for symmetric groups of small degrees. *J. Algebra 320* (2008), 680–707.

[8]   L. Dornhoff, *Group representation theory, Part II*. Dekker, 1972.

[9]   J. A. Green, On the indecomposable representations of a finite group. *Math. Zeitschrift 70* (1958/59), 430–445.

[10]  G. James, A. Kerber, *The representation theory of the symmetric group*, vol. 16 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1981.

[11]  G. D. James, Some counterexamples in the theory of Specht modules. *J. Algebra 46* (1977), 457–461.

[12]  G. D. James, *The representation theory of the symmetric groups*, vol. 682 of *Lecture Notes in Mathematics*. Springer, Berlin, 1978.

[13]  G. D. James, Trivial source modules for symmetric groups *Arch. Math. (Basle) 41* (1983), 294–300.

[14]  S. Martin, *Schur algebras and representation theory*, vol. 112 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1993.

[15]  J. Müller, R. Zimmermann, Green vertices and sources of simple modules of the symmetric group. *Arch. Math. (Basle) 89* (2007), 87–108.

[16]  G. E. Murphy, A new construction of Young's seminormal representation of the symmetric group. *J. Algebra 69* (1981), 287–297.

[17]  G. M. Murphy, On decomposability of some Specht modules for symmetric groups. *J. Algebra 66* (1980), 156–168.

[18]  G. M. Murphy, M. H. Peel, Vertices of Specht modules. *J. Algebra 86* (1984), 85–97.

[19]  J. B. Olsson, McKay numbers and heights of characters. *Math. Scand. 38* (1976), 24–42.

[20]  M. H. Peel, Specht modules and symmetric groups. *J. Algebra 36* (1975), 88–97.

[21]  W. Specht, Die irreduziblen Darstellugen der symmetrischen Gruppe. *Math. Zeitschrift 39* (1935), 696–711.

[22]  J. Thévenaz, *G-Algebras and modular representation theory*. Oxford mathematical monographs. Oxford University Press, 1995.

[23]  M. Wildon, Two theorems on the vertices of Specht modules. *Arch. Math. (Basel) 81* (2003), 505–511.

[24]  M. Wildon, *Modular representations of symmetric groups*. D. Phil Thesis, Oxford University, 2004.