Abstract

Let X be a character table of the symmetric group S_n. It is shown that unless $n = 4$ or $n = 6$, there is a unique way to assign partitions of n to the rows and columns of X so that for all λ and ν, $X_{\lambda\nu}$ is equal to $\chi^\lambda(\nu)$, the value of the irreducible character of S_n labelled by λ on elements of cycle type ν. Analogous results are proved for alternating groups, and for the Brauer character tables of symmetric and alternating groups.

Labelling the character tables of symmetric and alternating groups

Mark Wildon

April 23, 2006

1 Introduction

In 1957 Nagao [13] proved that if G is a finite group with a character table which differs from a character table of the symmetric group S_n only by a permutation of its rows and columns, then G is isomorphic to S_n. In this paper we consider a question naturally raised by Nagao's result. To state it, we must recall that the ordinary irreducible characters of S_n are canonically labelled by the partitions of n, and that this set also labels the conjugacy classes of S_n. Given partitions λ and ν of n, let $\chi^\lambda(\nu)$ be the value of the irreducible character of S_n labelled by λ on elements of cycle type ν.

Now suppose that one has discovered (for example, by applying Nagao's theorem) that a given square matrix is an unlabelled character table of the symmetric group S_n. We ask: when can one go further, and *uniquely* reconstruct the partitions labelling its rows and columns? The answer is given by the following theorem.

Theorem 1.1. *Let X be a character table of the symmetric group S_n. Unless $n = 4$ or $n = 6$, there is a unique way to assign partitions of n to the rows and columns of X so that $X_{\lambda\nu} = \chi^\lambda(\nu)$ for all partitions λ, ν of n. If $n = 4$ or $n = 6$ then there are exactly two different labellings.*

Probably the reader has already correctly guessed that the exception for $n = 6$ arises from the outer automorphism that exists only in this case. The exception for $n = 4$ appears to be a numerical coincidence. We discuss the two exceptional cases more fully in §2.1.

The main work begins in §2.2, where we show that, provided $n \geq 7$, there is only one possible way to assign partitions of the form $(n - m, m)$ to the rows of X. (We say that such partitions are *two-row partitions.*) Then in §2.3 we show that given such a partial row labelling, there is only one way to assign all the column labels. Of course, once we have fixed the column labels, the remaining row labels are uniquely determined. It follows that there is a unique way to label X. In §2.4 we give an efficient way to complete the row labelling. A question related to the way our proof links row and column labels is raised in §2.5.

It is natural to ask the analogous question for alternating groups, and for the Brauer character tables of symmetric and alternating groups. Strikingly, these questions may also be answered using the strategy we have just outlined, giving the results stated below in Theorem 3.1, Theorem 4.1 and Theorem 5.2 respectively.

In Theorem 5.2 on the Brauer character tables of alternating groups A_n, we only consider representations in odd characteristic. This is because the Clifford theory needed to relate simple representations of the alternating groups to simple representations of the symmetric groups is essentially the same in odd characteristic as in characteristic zero, and so the methods used in the characteristic zero case generalise easily. As A_n has index 2 in S_n, the situation in even characteristic is quite different, and so we do not attempt to deal with it here.

Finally we remark that an informal interpretation of our results is that the rich structure of representations of symmetric and alternating groups makes their character tables highly rigid. In §6 we make this idea more precise by putting our results in the general context of representations of finite groups. We also pose two open problems suggested by our work.

2 Proof of Theorem 1.1

2.1

We begin with S_6, the only symmetric group to have an outer automorphism (see for example [15, Theorem 7.9]). Any outer automorphism of S_6 permutes its conjugacy classes by

$$(6) \leftrightarrow (3,2),\ (3,3) \leftrightarrow (3),\ (2,2,2) \leftrightarrow (2)$$

and permutes its ordinary characters by

$$(5,1) \leftrightarrow (2,2,2),\ (2,1,1,1,1) \leftrightarrow (3,3),\ (4,1,1) \leftrightarrow (3,1,1,1).$$

Thus if σ and τ denote the corresponding permutations on the set of partitions of 6 then $\chi^\lambda(\nu) = \chi^{\lambda^\sigma}(\nu^\tau)$ for all partitions λ, ν. This gives two different ways to label the character table of S_6. Inspection of the table shows there are no more.

We now turn to S_4, which has the character table shown below.

	(1^4)	$(2,1,1)$	(4)	$(2,2)$	$(3,1)$
(4)	1	1	1	1	1
$(3,1)$	3	-1	1	-1	0
$(2,1,1)$	3	1	-1	-1	0
$(2,2)$	2	0	0	2	-1
(1^5)	1	-1	1	1	-1

Notice that if we swap the columns labelled by $(2,1,1)$ and (4), and the rows labelled by $(3,1)$ and $(2,1,1)$, we end up with the same matrix. Again it is easy to see that this is the only alternating way to label the character table. Unlike the case of S_6, this alternative labelling is not induced by any automorphism of the group. (See §6 for some remarks related to this phenomenon.)

Theorem 1.1 may readily be verified by inspection of the character tables if $n \leq 3$ or $n = 5$, so from now on we shall assume that $n \geq 7$.

2.2

Let $n \geq 7$ and let X be an unlabelled character table of S_n. We shall show that there is a unique way to assign two-row partitions (that is, partitions of the form $(n-m, m)$) to the rows of X.

By the orthogonality relations for ordinary characters, there is a unique row of X containing only positive entries. Similarly there is a unique column of X containing only positive entries. We may therefore uniquely identify the column corresponding to the identity element and the row corresponding to the trivial character, thus fixing the row label (n) and the column label (1^n).

For all $n \neq 6$ the symmetric group S_n has exactly two characters of degree $n-1$, namely $\chi^{(n-1,1)}$ and $\chi^{(2,1^{n-2})}$ (see [8] Theorem 2.4.10). These characters are defined by

$$\chi^{(n-1,1)}(g) = |\operatorname{Fix} g| - 1,$$

where $\operatorname{Fix} g$ is the set of elements fixed by g in its natural action on $\{1, \ldots, n\}$, and

$$\chi^{(2,1^{n-2})}(g) = \operatorname{sgn}(g)\chi^{(n-1,1)}(g).$$

The only elements in S_n with exactly $n-2$ fixed points are transpositions. Hence $\chi^{(n-1,1)}(t)$ takes $n-3$ as a value, while $\chi^{(2,1^{n-2})}$ can only take this value if $n-3 = 1 - |\operatorname{Fix} g|$ for some odd element $g \in S_n$. Clearly this can only happen if $n \leq 4$. We may therefore fix the row label $(n-1,1)$.

To proceed further, we make two key observations. First, given two rows of the unlabelled character table X we may multiply the corresponding entries and so obtain a new character; this is the character of the tensor product of the representations corresponding to the rows we multiplied. Second, still working only with the unlabelled table, we may take the inner product of our new character with each row of the table. The resulting sequence of non-negative integers tells us its irreducible constituents.

The following lemma gives the results we need to exploit these observations. (The main idea in the proof comes from [1].)

Lemma 2.1. *If $n \geq 4$ then*

$$\chi^{(n-1,1)}\chi^{(n-1,1)} = \chi^{(n)} + \chi^{(n-1,1)} + \chi^{(n-2,2)} + \chi^{(n-2,1,1)}.$$

If $r \geq 2$ and $n > 2r+1$ then

$$\begin{aligned}\chi^{(n-1,1)}\chi^{(n-r,r)} = \chi^{(n-r+1,r-1)} + \chi^{(n-r,r)} + \chi^{(n-r,r-1,1)} \\ +\chi^{(n-r-1,r+1)} + \chi^{(n-r-1,r,1)}.\end{aligned}$$

Proof. Recall that $\chi^{(n-1,1)} + \chi^{(n)}$ is the permutation character of S_n acting on $\{1,\ldots,n\}$. As this character is induced from the subgroup S_{n-1} of S_n, we have

$$\chi^{(n-1,1)}\theta = \left(1_{S_{n-1}}\uparrow^{S_n}\right)\theta - \theta = \theta\downarrow_{S_{n-1}}\uparrow^{S_n} - \theta$$

for any character θ of S_n. The result now follows from the branching rule for ordinary representations of S_n (see [9, Ch. 9]). □

The first part of this lemma implies that by decomposing the product $\chi^{(n-1,1)}\chi^{(n-1,1)}$ we may find the two rows of X which should be labelled by $\chi^{(n-2,2)}$ and $\chi^{(n-2,1,1)}$. Since these irreducible characters have different degrees (see Lemma 2.2 below) we may fix the row labels $(n-2,2)$ and $(n-2,1,1)$.

The remaining two-row characters are found in a similar way. Suppose inductively that there is a unique way to assign the labels $(n-s,s)$ and $(n-s,s-1,1)$ for $s \leq r$. If $2r = n$ or $2r+1 = n$ then we are finished, so we assume that $n > 2r+1$. By decomposing the product $\chi^{(n-1,1)}\chi^{(n-r,r)}$ we may find the two rows which should be labelled by $\chi^{(n-r-1,r+1)}$ and $\chi^{(n-r-1,r,1)}$. Again by Lemma 2.2, these characters have different degrees, so we may fix the row labels $(n-r-1,r+1)$ and $(n-r-1,r,1)$. This completes the inductive step.

Lemma 2.2. *If $r \geq 2$ and $n \geq 2r$ then*

$$\chi^{(n-r,r)}(1) < \chi^{(n-r,r-1,1)}(1).$$

Proof. By the hook-formula for the irreducible character degrees of S_n (see [8, Theorem 2.3.21]),

$$\chi^{(n-r,r)}(1) = \binom{n}{r}\frac{n-2r+1}{n-r+1}$$

which is less than

$$\chi^{(n-r,r-1,1)}(1) = (r-1)\binom{n}{r}\frac{n-2r+2}{n-r+2}.$$ □

2.3

Fix a column of our unlabelled character table X. Suppose that in one possible labelling of the columns of X, the corresponding conjugacy class has $g \in S_n$ as a representative. We shall show that the cycle type of g can be reconstructed from the character values $\chi^{(n-r,r)}(g)$, which are known from the previous section. This shows that the column labels of X are uniquely determined.

Since $\chi^{(n-1,1)}(g) = |\operatorname{Fix} g| - 1$, we can find the number of fixed points of g. Suppose inductively that we know that g has a_1 fixed points, a_2 2-cycles, and so on, up to a_{r-1} $(r-1)$-cycles, where $1 < r \leq n/2$. Let

$$\pi^{(n-r,r)} = \sum_{s=0}^{r} \chi^{(n-s,s)}.$$

This is the permutation character of S_n acting on r-subsets of $\{1, \ldots, n\}$. Using only known character values we may calculate $\pi^{(n-r,r)}(g)$, and so find the number of r-subsets fixed by g. This equals the number of r-cycles in g, plus some further quantity that can be computed given $a_1, \ldots, a_{r-1}$. Therefore a_r may be determined. This gives us the number of cycles of all lengths $r \leq n/2$. Since g has at most one cycle of greater length, this gives enough information to determine its cycle type.

2.4

Now we have found all the column labels of X, the remaining row labels are uniquely determined. Here we briefly give a practical way to determine these labels.

It follows from a result of Kramer [11] that two irreducible characters of S_n which agree on every cycle of length r for $1 \leq r \leq n$ are equal. (See [19] for a short proof of this, and some related results.) Suppose then that we have already computed the character values $\chi^\lambda(n-r, 1^r)$ for $0 \leq r < n$ and λ any partition of n. One efficient way to do this is to use the Murnagham–Nakayama rule (see [9, Ch. 21]) and the hook-formula. Then to find the partition labelling row i of the character table X, we need only compare the values $X_{i\nu}$ for $\nu = (n-r, 1^r)$ and $0 \leq r < n$ with those in our pre-computed table.

2.5

It is natural to ask when it is possible to identify the label of a symmetric group character or conjugacy class just from the multiset of values in the corresponding row or column in the character table. For conjugacy classes there are many cases where this is impossible; the following proposition gives two families of examples.

Proposition 2.3. *Let $m \geq 4$.*

(i) *The conjugacy classes $(2m-2, 2)$ and $(2m-2, 1, 1)$ of S_{2m} have the same multiset of character values.*

(ii) *The conjugacy classes $(2m-3, 4)$ and $(2m-3, 2, 1, 1)$ of S_{2m+1} have the same multiset of character values.*

Proof. The character values required may easily be computed using the Murnagham–Nakayama rule. In (i), the classes take $+1$ and -1 each with multiplicity $2m-2$, and all other values are 0. In (ii), the classes take $+1$ and -1 each with multiplicity $4m-6$ and all other values are 0. $\square$

I do not know of any example of two different characters of S_n (for $n \neq 4, 6$) which have the same multiset of values. A search with the computer algebra package GAP [6] shows this behaviour does not occur for $n \leq 30$, $n \neq 4, 6$. It is therefore still possible that a slightly stronger result than our Theorem 1.1 is true.

3 Alternating groups

The outer automorphisms of the alternating group A_n induced by the conjugacy action of S_n lead to some inevitable ambiguity in the row and column labels of its character table. Any outer automorphism of A_n acts as an involution, swapping pairs of split characters and conjugacy classes.

More precisely, if λ is a partition of n, then some basic Clifford theory shows that $\chi^\lambda\downarrow_{A_n}$ is reducible if and only if

$$\chi^\lambda = \chi^\lambda \times \mathrm{sgn}$$

which holds if and only if λ is self-conjugate; in this case χ^λ splits as a sum of two irreducible characters of A_n. Similarly, the conjugacy class labelled by the partition ν of n splits in A_n if and only if ν has odd distinct parts. We distinguish split characters and classes by arbitrarily allocating $+$ and $-$ signs. Whichever allocation we choose, it will be reversed under the outer action of S_n.

For example, the character table of A_5 is

	(1^5)	$(2,2,1)$	$(3,1,1)$	$(5)^+$	$(5)^-$
(5)	1	1	1	1	1
$(4,1)$	4	0	1	-1	-1
$(3,2)$	5	1	-1	0	0
$(3,1,1)^+$	3	-1	0	α	β
$(3,1,1)^-$	3	-1	0	β	α

where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5}/2)$. An alternative labelling in which the characters and conjugacy classes labelled $+$ and $-$ are swapped is given by the action of S_5.

The next theorem states that swapping signs usually gives all possible labellings. Thus if there are s split conjugacy classes then there are usually 2^s different labellings.

Theorem 3.1. *Let X be a character table of the alternating group A_n. Provided $n \neq 6$ there is a unique way to assign non-self-conjugate partitions to the rows of X and partitions not having odd distinct parts to the columns of X so that $X_{\lambda\nu} = \chi^\lambda(\nu)$ for all such λ and ν. The labels of the split characters and conjugacy classes are uniquely determined up to signs.*

Proof. For $n \leq 5$ the theorem can be readily verified by inspecting the tables. By [8, Theorem 2.5.15], provided $n \geq 7$, the only character of A_n of degree $n-1$ is the one labelled by $(n-1, 1)$. So we can fix this label. Since no self-conjugate partitions appear in the calculations of §2.2, the remaining two row characters may then be identified as before. The cycle types of the columns may also be identified as before.

The character values of the split conjugacy classes are described by [4, Proposition 5.3]. It follows from this proposition that the labels $+$ and $-$ may be assigned independently on different split classes. Once all the column labels are fixed, there is then a unique way to label the rows. □

Inspection of the character table of A_6 shows that it has 4 different labellings. Using the definitions given in §6 below, we have $\mathrm{cAut}(A_6) \cong \chi\mathrm{Aut}(A_6) \cong \mathrm{Out}(A_6) \cong \langle (12), (34) \rangle$.

4 Brauer character tables of symmetric groups

Let F be a field of prime characteristic p. Recall that a partition is said to be *p-regular* if it has at most $p-1$ parts of any given size. It is proved in [9, Ch. 11] that the irreducible FS_n representations are parametrised by the p-regular partitions of n. Let D^μ be the p-modular irreducible corresponding to the p-regular partition μ and let ϕ^μ be the Brauer character of D^μ. (See [14] for an introduction to Brauer characters.) If ν is a partition of n with no part divisible by p then we write $\phi^\mu(\nu)$ for the value of ϕ^μ on the conjugacy class of p'-elements labelled by ν.

In this section we shall prove the following theorem.

Theorem 4.1. *Let X be a Brauer character table of the symmetric group S_n in characteristic p. Unless $n = 6$, or $n = 4$ and $p > 2$, there is a unique way to assign p-regular partitions to the rows of X and partitions with no part divisible by p to the columns of X so that $X_{\mu\nu} = \phi^\mu(\nu)$ for all such μ and ν. In the exceptional cases there are exactly 2 different labellings.*

To prove this theorem we shall need some further results on the modular representations of symmetric groups. Recall that the *decomposition matrix* $D_n(p)$ of S_n in characteristic p is the matrix defined by

$$\chi^\lambda = \sum_\mu D_n(p)_{\lambda\mu}\phi^\mu$$

where λ is a partition of n and the sum is over all p-regular partitions μ. (Here and elsewhere when we write a relation between ordinary and Brauer characters it is intended to hold for p'-elements only. This abuse of notation will not lead to any ambiguity for us.)

We also need the dominance order on partitions. Recall that if λ and μ are partitions of the same number then we say that μ *dominates* λ, and write $\mu \trianglerighteq \lambda$ if

$$\mu_1 + \mu_2 + \ldots + \mu_j \geq \lambda_1 + \lambda_2 + \ldots + \lambda_j \quad \text{for all } j \geq 1.$$

where if j exceeds the number of parts of μ we set $\mu_j = 0$, and similarly for λ. The following lemma is Corollary 12.3 in [9].

Lemma 4.2. *Let λ be a partition of n and let μ be a p-regular partition of n. If $D_p(n)_{\lambda\mu} \neq 0$ then $\mu \trianglerighteq \lambda$. Moreover $D_p(n)_{\lambda\lambda} = 1$.* □

Finally we need a simple branching rule for modular representations.

Lemma 4.3. *Let $\mu = (\mu_1, \ldots, \mu_k)$ be a partition of n such that $\mu_1 > \mu_2$. If $\bar{\mu} = (\mu_1 - 1, \mu_2, \ldots, \mu_k)$ is p-regular then $\phi^\mu\downarrow_{S_{n-1}} = \phi^{\bar{\mu}} + \psi$ where ψ is a sum of Brauer characters ϕ^λ labelled by partitions λ such that $\lambda \triangleright \bar{\mu}$.*

Proof. In future we shall write $\psi \triangleright \mu$ if ψ is an integral linear combination of Brauer (or ordinary) characters labelled by partitions λ such that $\lambda \triangleright \mu$. By Lemma 4.2 we have $\phi^\mu = \chi^\mu - \theta$ where $\theta \triangleright \mu$. It then follows from the ordinary branching rule that

$$\phi^\mu\big\downarrow_{S_{n-1}} = \chi^{\bar{\mu}} + \psi$$

where $\psi \triangleright \bar{\mu}$. Now apply Lemma 4.2 one more time. □

We can now begin the proof of Theorem 4.1. We follow as closely as possible the method of proof used in §2; thus §4.1 below is the analogue of §2.1, and so on.

4.1

If $n \leq 6$ the theorem may readily be verified by inspecting the tables. (Brauer character tables for $p = 2$ and $p = 3$ and $n \leq 10$ appear in Appendix I.F of [8]; for $p = 5$ and $n = 5, 6$ the required tables may easily be calculated by hand, as all blocks have weight 0 or 1.) The only difference in behaviour from the ordinary case occurs when $n = 4$: the Brauer character table of S_4 in characteristic 2 is shown below.

	(1^4)	$(3,1)$
(4)	1	1
$(3,1)$	2	-1

Clearly there is no longer any ambiguity about the labels.

4.2

Let $n \geq 7$ and let X be an unlabelled Brauer character table of S_n in characteristic p. Although we can no longer exploit row and column orthogonality, it is still easy to assign the row label (n) and the column label (1^n).

It was first proved by Wagner (see [16, 17]) that if S is a simple FS_n-module with $\dim S \leq n-1$ then, provided $n \geq 7$, either S is 1-dimensional, or S is isomorphic to one of $D^{(n-1,1)}$ or $D^{(n-1,1)} \otimes \mathrm{sgn}$. (Of course if $p = 2$ then these representations are the same. For an alternative shorter proof see James [10, Theorem 6].) The Brauer character of $D^{(n-1,1)}$ is

$$\phi^{(n-1,1)}(g) = |\operatorname{Fix} g| - \begin{cases} 1 & \text{if } p \nmid n \\ 2 & \text{if } p \mid n. \end{cases}$$

If p is odd and $p \nmid n$ then, of $\phi^{(n-1,1)}$ and $\phi^{(n-1,1)}\,\mathrm{sgn}$, only $\phi^{(n-1,1)}$ takes the value $n-3$. Similarly, if p is odd and $p \mid n$ then only $\phi^{(n-1,1)}$ takes the value $n-4$. Hence in all cases we may identify the row of X labelled by $(n-1,1)$.

We are now in a position to identify the rows of X labelled by all two-row partitions. As before, we do this inductively by taking products of characters. However, as there is no simple formula for the degrees of the characters $\phi^{(n-r,r)}$, our approach has to be slightly more subtle. The following lemma is the analogue of Lemma 2.1.

Lemma 4.4. *If $r \geq 1$ and $n > 2r+1$ then*

$$\phi^{(n-1,1)}\phi^{(n-r,r)} = b\phi^{(n-r-1,r+1)} + \phi^{(n-r-1,r,1)} + \psi$$

where $b \geq 1$ and ψ is an integral linear combination of irreducible Brauer characters labelled by partitions μ such that $\mu \trianglerighteq (n-r, r-1, 1)$.

Proof. Since $\phi^{(n-1,1)} = 1_{S_{n-1}}\!\uparrow^{S_n} - c\phi^{(n)}$ where $c \in \{1,2\}$, it is sufficient to show that

$$\phi^{(n-r,r)}\big\downarrow_{S_{n-1}}\big\uparrow^{S_n} = b\phi^{(n-r-1,r+1)} + \phi^{(n-r-1,r,1)} + \psi$$

where $b \geq 1$ and $\psi \trianglerighteq (n-r, r-1, 1)$. We have

$$\phi^{(n-r,r)} = \chi^{(n-r,r)} - \theta$$

where $\theta \trianglerighteq (n-r+1, r-1)$. Hence by the the ordinary branching rule,

$$\phi^{(n-r,r)}\big\downarrow_{S_{n-1}}\big\uparrow^{S_n} = \chi^{(n-r-1,r+1)} + \chi^{(n-r-1,r,1)} + \psi$$

where $\psi \trianglerighteq (n-r, r-1, 1)$. Now apply Lemma 4.2. $\square$

For simplicity we state and prove the following proposition for $p > 2$, and explain later the small modifications needed when $p = 2$.

Proposition 4.5. *Let $n \geq 7$ and let C be a labelled Brauer character table of S_n in characteristic $p > 2$. Suppose that the first column is labelled by (1^n), and that the rows are arranged so that the first row is labelled by (n), the second by $(n-1, 1)$ and the next $2(r-1)$ with the labels*

$$(n-2,2), (n-3,3), \ldots, (n-r, r)$$
$$(n-2,1^2), (n-3,2,1), \ldots, (n-r, r-1, 1).$$

in any order. *If we are given the first $2r$ rows of C with the row and column labels removed, then the row labels may be uniquely reconstructed.*

Proof. We work by induction on n. If $n = 7$ and $p = 3$ or $p = 5$ then the Brauer characters of S_n that can appear in X have distinct degrees, so the result is immediate. If $p = 7$ then both $\phi^{(5,2)}$ and $\phi^{(4,3)}$ have degree 14, but only the former takes 6 as a value. (As both these characters lie in blocks of weight zero this can be seen directly from the ordinary character table.)

Suppose now that $n \geq 8$. By hypothesis, we may immediately attach the row labels (n) and $(n-1, 1)$ to C. We now attempt to reach a situation in which the inductive hypothesis for $n-1$ can be applied. Notice first that the values of $\phi^{(n-1,1)}$ determine which columns in the table come from conjugacy classes with at least one fixed point, and so are relevant when we restrict a character to S_{n-1}. The restriction of $\phi^{(n)}$ to S_{n-1} is, of course, $\phi^{(n-1)}$. We may obtain $\phi^{(n-2,1)}$ by removing any copies of $\phi^{(n-1)}$ from $\phi^{(n-1,1)}\downarrow_{S_{n-1}}$.

By Lemma 4.4 above, when we express $\phi^{(n-1,1)}\phi^{(n-1,1)}$ as a sum of rows of X, two new characters appear: $\phi^{(n-2,2)}$ and $\phi^{(n-1,1,1)}$. When we restrict these new characters to S_{n-1} we get, in addition to any copies of $\phi^{(n-2,1)}$ and $\phi^{(n-1)}$ that may be present, two new Brauer characters of S_{n-1}: namely $\phi^{(n-3,1,1)}$ and $\phi^{(n-3,2)}$. We may now apply the inductive hypothesis (with $r = 2$) to determine which label should go with which. To get back to S_n we use Lemma 4.3. Together with Lemma 4.2, it implies that $\phi^{(n-2,2)}\downarrow_{S_{n-1}}$ does not contain $\phi^{(n-3,1,1)}$, whereas $\phi^{(n-2,1,1)}\downarrow_{S_{n-1}}$ does. We use this to fix the labels $(n-2, 2)$ and $(n-2, 1, 1)$.

The remaining two-row labels are fixed by repeating this argument, in a way closely analogous to the proof of Lemma 2.2. We therefore leave the remaining details of the proof to the reader. $\square$

By a further induction on r we may use this proposition to identify the rows of X labelled by two-row partitions.

If $p = 2$ then the statement of Proposition 4.5 must be slightly modified. We must delete $(n-2, 1, 1)$ from the list as it is no longer p-regular, and if n is even then we must also delete $(n/2, n/2)$. The main change in the proof is that now $\phi^{(n-1,1)}\phi^{(n-1,1)}$ only contains one new Brauer character, $\phi^{(n-2,2)}$; this makes the first step slightly simpler. After that, no alterations are needed, unless $n = 2m$ is even, in which case the last two row Brauer character we must find is $\phi^{(m+1,m-1)}$. Again this makes the process slightly simpler.

4.3

We now determine the column labels of X. By §4.2 we may rearrange the rows of X so that the row labelled (n) appears first, followed by the rows labelled by two row partitions in the order given by the dominance order. We also order the rows and columns of the decomposition matrix $D_n(p)$ by the dominance order. By Lemma 4.2, $D_n(p)_{\lambda\mu} = 0$ if λ has at most two rows and μ does not. Hence, the matrix $D_n(p)X$ has, at its top, the values of the characters χ^λ for partitions λ with at most two rows.

We can now use the same argument as in the ordinary case to determine the column labels of X. Once we have a complete set of column labels the row labels are, of course, fixed. This completes the proof of Theorem 4.1.

5 Brauer character tables of alternating groups

Provided we work in odd characteristic, the modifications to the work of §4 that are needed to deal with the Brauer character tables of alternating groups are analogous to the modifications we needed to the work of §2 to deal with the ordinary character tables of alternating groups.

Let X be an unlabelled Brauer character table of A_n in odd characteristic p. If λ is a p-regular partition of n, let $m(\lambda)$ be the p-regular partition of n defined by

$$\phi^\lambda \times \mathrm{sgn} = \phi^{m(\lambda)}.$$

(The map m was first considered by Mullineux in [12]). By the Clifford theory given in [5], the Brauer character ϕ^λ splits on restriction to A_n if and only if $\lambda = m(\lambda)$. As before, we label split characters and conjugacy classes by $+$ and $-$ signs.

As usual, to get started we need to identify the character labelled by $(n-1, 1)$. For this we use Theorem 1.1 in [17], which states that if $n \geq 7$ and ϕ is an odd characteristic Brauer character of A_n such that $\phi(1) \leq n$ then $\phi = \phi^{(n-1,1)}\downarrow_{A_n}$. (For an alternative shorter proof of this result for

$n \geq 10$ see [10, Theorem 7(ii)]; the result can easily be checked directly in the remaining cases.)

We also need to know that none of the characters considered in §4.2 split on restriction to A_n. For this to hold we must take $n \geq 8$.

Lemma 5.1. *Let $n \geq 8$ and let λ be a partition of n. If either λ has at most two rows, or λ is of the form $(n-r, r-1, 1)$ where $2 \leq r \leq n/2$ then $\lambda \neq m(\lambda)$. Hence ϕ^λ does not split on restriction to A_n.*

Proof. It follows easily from Ford's description [5] of the Mullineux map that $\lambda \neq m(\lambda)$ if either of the conditions on λ hold. □

It is not possible to take $n \geq 7$ (as was the case in §4.2) because if $p = 3$ then $m((4,2,1)) = (4,2,1)$. The base case in the analogue of Proposition 4.5 is therefore $n = 8$. Calculation shows that the Brauer characters of A_8 that can appear in the table C have distinct degrees when $p = 3$ and when $p = 7$. If $p = 5$ then $\phi^{(6,1,1)} = \phi^{(5,3)} = 21$, but only the former character takes 6 as a value, so again there is no ambiguity. Thus we may identify the rows of X labelled by two-row partitions. The column labels may now be determined in essentially the same way as §4.3.

Direct examination of the cases for $n \leq 7$ gives the following theorem.

Theorem 5.2. *Let X be a Brauer character table of the alternating group A_n in odd characteristic p. Provided $n \neq 6$ there is a unique way to assign p-regular partitions λ such that $m(\lambda) \neq \lambda$ to the rows of X, and partitions not all of whose parts are odd, and with no part divisible by p, to the columns of X, so that $X_{\lambda\mu} = \phi^\lambda(\mu)$ for all such λ and μ. The labels of the split characters and conjugacy classes are uniquely determined up to signs.* □

When $n = 6$ and $p = 3$ there are two different labellings, interchanged by the conjugacy action of S_6. When $n = 6$ and $p = 5$ there are again two different labellings, but this time S_6 acts trivially, and they are interchanged only by the outer automorphism of S_6.

6 A more general setting

Given a arbitrary finite group G, there is usually no canonical way to label the rows and columns of its character table. However, this need not stop us from considering analogous versions of our results.

Let k be the number of conjugacy classes of G, and let X be a character table of G. We say that a pair $(\sigma, \tau) \in S_k \times S_k$ is an *automorphism* of X if $X_{i\sigma, j\tau} = X_{ij}$ whenever $1 \leq i, j \leq k$. Let $\mathrm{Aut}(X)$ be the group of all automorphisms of X. It is clear that for each $\sigma \in S_k$ there is at most one $\tau \in S_k$ such that $(\sigma, \tau) \in \mathrm{Aut}(X)$. We may therefore define a group $\chi\mathrm{Aut}(G)$ by

$$\chi\mathrm{Aut}(G) = \{\sigma \in S_k : (\sigma, \tau) \in \mathrm{Aut}(X) \text{ for some } \tau \in S_k\}.$$

This group is well-defined up to conjugacy in S_k. For example, our Theorem 1.1 states that $\chi\mathrm{Aut}(S_n)$ is trivial unless $n = 4$ or $n = 6$.

An interesting property of the elements of $\chi\mathrm{Aut}(G)$ is that if $(\sigma, \tau) \in \chi\mathrm{Aut}(G)$ then σ and τ have the same cycle structure. This result, usually known as Brauer's Permutation Lemma, was proved by Brauer in [2, §6].

Problem 6.1. *Calculate* $\chi\mathrm{Aut}(G)$ *for important classes of groups.*

In connection with this problem, it is useful to explore the relationship between $\chi\mathrm{Aut}(G)$ and $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$, the group of outer automorphisms of G. Clearly there is a group homomorphism

$$c : \mathrm{Out}(G) \to \chi\mathrm{Aut}(G)$$

defined for $\gamma \in \mathrm{Out}(G)$ by letting $c(\gamma)$ be the permutation induced by γ on the ordinary characters of G. In some cases c is an isomorphism — for example, this is the case if G is abelian, or G is a symmetric group other than S_4. But, as the example of S_4 shows, c need not be surjective. (Indeed, since for $n \geq 9$ there are always at least two self-conjugate partitions of n, the alternating groups A_n give an infinite family of examples in which c is not surjective.)

The dual question of whether c must be injective, or equivalently, whether there is a finite group G and an outer automorphism $\gamma \in \mathrm{Aut}(G)$ such that $\chi^\gamma = \chi$ for all irreducible characters χ, was considered by Burnside: see Note B in [3]. In [18], G. E. Wall gives an example in which G has order 32 and γ has order 4.

It is worth noting that Burnside's question can be stated without even mentioning characters, since by Brauer's permutation lemma, t is such an automorphism if and only if t permutes within themselves all the conjugacy classes of G. (Incidentally, it seems clear from [3, §217] that Brauer's permutation lemma was already well known to Burnside.)

Another obvious question, which is related to Problem 6.1, is:

Problem 6.2. *Is the map* $c : \mathrm{Out}(G) \to \chi\mathrm{Aut}(G)$ *always injective when* G *is a finite simple group?*

A third problem, which can be answered more easily, arises from the definition of $\chi\mathrm{Aut}(G)$. If we look instead at the admissible *column* permutations of X then we obtain the group

$$\mathrm{cAut}(G) = \{\tau \in S_k : (\sigma, \tau) \in \mathrm{Aut}(X) \text{ for some } \sigma \in S_k\}.$$

By Brauer's permutation lemma the groups $\chi\mathrm{Aut}(G), \mathrm{cAut}(G) \leq S_k$ are isomorphic as abstract groups, *via* an isomorphism preserving the cycle types of elements. But this on its own does not guarantee that they are permutation isomorphic, as the two subgroups of S_6,

$$\langle (12)(34), (13)(24) \rangle\,, \quad \langle (12)(34), (12)(56) \rangle$$

show. (There are many more examples of this type.) The following example shows that $\chi\mathrm{Aut}(G)$ and $\mathrm{cAut}(G)$ need *not* be permutation isomorphic, and so we made a genuine choice in concentrating on $\chi\mathrm{Aut}$ earlier.

Example 6.3. *Let $G \cong C_2 \times D_8$, where D_8 is the dihedral group of order 8. The character table of G is, with one ordering of the rows and columns:*

$$\begin{array}{cccccccccc}
-1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 \\
-1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\
-1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 2 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 & 2 & -2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}$$

One finds that

$$\begin{aligned}
\chi\mathrm{Aut}(G) &= \langle (1234)(56), (34)(57), (89) \rangle\,, \\
\mathrm{cAut}(G) &= \langle (1234)(56), (25)(46), (78) \rangle
\end{aligned}$$

where the isomorphism $\chi\mathrm{Aut}(G) \cong \mathrm{cAut}(G)$ is indicated by the order of generators. As the orbits of $\chi\mathrm{Aut}(G)$ have sizes $4, 3, 2, 1$ whereas the orbits of $\mathrm{cAut}(G)$ have sizes $6, 2, 1, 1$, the two groups are not permutation isomorphic. (Abstractly, both are isomorphic to $S_4 \times C_2$.)

Finally we mention a theorem of Higman (see [7, Theorem 8.21]) which states that given a character table of a finite group, one can determine prime divisors of the orders of the group elements corresponding to any given column. It is well known that the dihedral and quaternion groups of order 8 have the same character table, so this is the most one can hope for in general. Theorem 1.1 and Theorem 3.1 imply that for symmetric and alternating groups much more is true.

Corollary 6.4. *Given an unlabelled character table of a symmetric group other than S_4 one may determine the order of the elements corresponding to any of its columns. The same result holds for any alternating group.* □

Acknowledgements

I should like to thank Dr Christine Bessenrodt for the helpful comments she made when this work was beginning and the anonymous referee for his or her valuable suggestions.

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