MULTIPLICITY-FREE REPRESENTATIONS OF SYMMETRIC GROUPS

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ABSTRACT. Building on work of Saxl, we classify the multiplicity-free permutation characters of all symmetric groups of degree 66 or more. A corollary is a complete list of the irreducible characters of symmetric groups (again of degree 66 or more) which may appear in a multiplicity-free permutation representation. The multiplicity-free characters in a related family of monomial characters are also classified. We end by investigating a consequence of these results for Specht filtrations of permutation modules defined over fields of prime characteristic.

Keywords: multiplicity-free, symmetric group, permutation character, monomial character, Specht filtration

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In this paper we prove three theorems on the multiplicity-free representations of symmetric groups. These theorems have interesting consequences for the permutation actions of symmetric groups, and for the theory of Specht filtrations of permutation modules, while also being of interest in their own right.

Our notation is standard. Let S_n denote the symmetric group of degree n, and let χ^λ denote the ordinary irreducible character of S_n canonically labelled by the partition λ of n. (For an account of the character theory of the symmetric group see Fulton & Harris [4, Chapter 4], or James [12]. We shall use James' lecture notes as the main source for the deeper results we need.) A character π of S_n is said to be *multiplicity-free* if $\langle \pi, \chi^\lambda \rangle \leq 1$ for all partitions λ of n. If θ is a character of a subgroup of S_n then we write $\theta\uparrow^{S_n}$ for the character of S_n induced by θ. Dually, the arrow $\downarrow$ denotes restriction. Later we shall extend this notation from characters to their associated representations. If H is a subgroup of S_l then $H \wr S_m$ is the wreath product of H with S_m, acting as a subgroup of S_{lm} (as defined in [3, §1.10]). Finally, let A_n denote the alternating group of degree n.

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Our first two theorems are motivated by a result of Inglis, Richardson and Saxl [10] which shows that every irreducible character of a symmetric group is a constituent of a multiplicity-free monomial character.

Theorem (Inglis, Richardson, Saxl)**.** *Let $m \in \mathbf{N}$ and let t be a fixed-point-free involution in the symmetric group S_{2m}. If $n = 2m + f$ where $f \in \mathbf{N}_0$ then*
$$1_{C_{S_{2m}}(t)} \times \mathrm{sgn}_{S_f} \big\uparrow_{C_{S_{2m}}(t) \times S_f}^{S_n} = \sum_{\lambda} \chi^\lambda$$
where the sum is over all partitions λ of n with precisely f odd parts.

Our Theorem 1 shows that conversely, these characters are nearly the only ones of their type that are multiplicity-free.

Theorem 1. *Let $n \geq 7$, let $k \leq n$ and let $x \in S_k$ be a fixed-point-free permutation. Let θ be a 1-dimensional character of S_{n-k}. The monomial character*
$$\psi = \big(1_{C_{S_k}(x)} \times \theta\big)\big\uparrow_{C_{S_k}(x) \times S_{n-k}}^{S_n}$$
is multiplicity-free if and only if either *x is a 2-cycle and $k = 2$* or *x is a 3-cycle and $k = 3$* or *x is a fixed-point-free involution in S_k and $\theta = \mathrm{sgn}_{S_{n-k}}$.*

We prove this theorem in §2 below. The proof is straightforward, and will help to introduce the techniques used in the remainder of the paper.

In light of the theorem of Inglis, Richardson and Saxl, it is very natural to ask whether every irreducible character of a symmetric group is a constituent of a multiplicity-free *permutation* character. Our second theorem, which builds on work of Saxl [15], gives the classification needed to show that this is very far from the case.

Theorem 2. *Let $n \geq 66$. The permutation character of S_n acting on the cosets of a subgroup G is multiplicity-free if and only if one of:*

(a1) *$A_n \leq G \leq S_n$ or $A_{n-1} \leq G \leq S_{n-1}$ or $G = A_{n-2} \times S_2$ or $G = S_{n-2} \times S_2$ or $G = A_n \cap (S_{n-2} \times S_2)$;*

(a2) *$A_{n-k} \times A_k \leq G \leq S_{n-k} \times S_k$ where $3 \leq k < (n-1)/2$;*

(b1) *$n = 2k$ and $A_k \times A_k < G \leq S_k \wr S_2$;*

(b2) *$n = 2k+1$ and either $A_{k+1} \times A_k < G \leq S_{k+1} \times S_k$ or G is a subgroup of $S_k \wr S_2$ of index ≤ 2 other than $S_k \times S_k$;*

(c) *$n = 2k$ or $n = 2k+1$ and $G = S_2 \wr S_k$;*

(d) *$n = 2k$ where k is odd and $G = A_{2k} \cap (S_2 \wr S_k)$;*

(e) *$G = G_k \times A_{n-k}$ or $G = G_k \times S_{n-k}$ where k is either 5, 6 or 9, or $G = A_n \cap (G_k \times S_{n-k})$ where $k = 5$ or 6, and G_5 is the Frobenius group of order 20 acting on 5 points, G_6 is $\mathrm{PGL}(2,5)$ in its natural projective action on 6 points and G_9 is $\mathrm{P\Gamma L}(2,8)$ in its natural projective action on 9 points.*

It seems unavoidable that cases (a) and (b) have a slightly fiddly statement. The part of the proof that leads to these cases is, however, the most routine. The reader may wish to refer ahead to Figure 2 in §2.2 which shows the subgroups of $S_k \wr S_2$ that appear in case (b).

If $n = 2k$ then the subgroup $S_2 \wr S_k$ in case (c) is the centralizer of a fixed-point-free involution in S_n. It may also be helpful to recall that $S_2 \wr S_k \leq S_{2k}$ is permutation isomorphic to the Weyl group of type B_k in its action on the vectors $\{\pm\epsilon_1, \ldots, \pm\epsilon_k\}$ spanning the root space for $\mathsf{so}(2k+1, \mathbf{C})$. (See [9, Chapter 3] for more details.) Under the natural embedding $\mathsf{so}(2k, \mathbf{C}) \to \mathsf{so}(2k+1, \mathbf{C})$, the Weyl group of type D_k acts on $\{\pm\epsilon_1, \ldots, \pm\epsilon_k\}$ as a subgroup of index 2 in B_k; it is this subgroup which appears in case (d) of Theorem 2. We define the subgroups in case (e) more fully in §2.3 below.

A corollary of Theorem 2 is a complete list of the irreducible characters of S_n for $n \geq 66$ which may appear in a multiplicity-free permutation representation (see Corollary 8 below). The reader will see that there are very few such characters; in particular, if $n \geq 20$, then $\chi^{(n-4,3,1)}$ never appears in such a representation.[1] Since a permutation character of a symmetric group is multiplicity-free if and only if all the orbitals in the corresponding permutation action are self-paired (see [3, page 46]), Theorem 2 also serves to classify such actions.

While Theorem 2 is stated for $n \geq 66$, the proof given in §2 below gives a complete classification for all $n \geq 20$. The predicted list of subgroups has been checked using the computer algebra package MAGMA [2] to be correct for $20 \leq n \leq 23$. The same check has been made for the list of irreducible characters in Corollary 8. One could easily use MAGMA to generate the full list of subgroups of symmetric groups of degree < 20 which give multiplicity-free permutation characters, but we shall not pursue this possibility here. The author recently learned of parallel work by C. Godsil and K. Meagher [**?**], to appear in Annals of Combinatorics. Their paper gives a complete classification of multiplicity-free permutation characters of every degree. When $n \geq 20$, their results are in agreement with Theorem 2.

Our third theorem concerns the permutation modules whose ordinary characters appear in Theorem 2. Note that, by [3, Theorem 3.5], these are exactly the permutation modules whose centralizer algebra is abelian. To understand the statement of this theorem the reader will need to know a little about Specht modules: see [12, Chapters 4, 5] for an introduction. We recall here that if $S^\lambda_{\mathbf{Z}}$ is the integral Specht module for $\mathbf{Z}S_n$ labelled by the partition λ of n then, regarding the entries of the representing matrices

[1] This result, together with Theorem 1, was first stated and proved in the author's D. Phil thesis [17, Chapter 4].

as rational numbers, $S^\lambda_{\mathbf{Z}}$ affords the ordinary irreducible character χ^λ. If, instead, we regard the entries as elements of a field F of prime characteristic, then the resulting module for FS_n, denoted simply S^λ, is usually no longer irreducible—indeed, determining the composition factors of Specht modules in prime characteristic is one of the main unsolved problems in modular representation theory.

We say that an FS_n-module U has a *Specht filtration* if there is a chain of submodules

$$0 \subset U_1 \subset \cdots \subset U_r \subset U$$

such that each successive quotient is isomorphic to a Specht module.

Theorem 3. *Let F be an algebraically closed field of prime characteristic $p > 3$ and let $n \geq 66$. If G is a subgroup of S_n such that the ordinary permutation character $1_G{\uparrow}^{S_n}$ is multiplicity-free, then each summand of the permutation module*

$$F{\uparrow}_G^{S_n}$$

is a self-dual module with a Specht filtration. The Specht module S^λ for FS_n appears in a Specht filtration of $F{\uparrow}_G^{S_n}$ if and only if χ^λ is a constituent of the ordinary character $1_G{\uparrow}^{S_n}$.

The author first suspected the existence of this theorem after reading Paget's paper [14]. Paget's main result is that the permutation modules coming from case (c) of Theorem 2 have a Specht filtration, with the expected Specht factors. It is a simple matter to adapt her work to deal with case (d).

In §4 we show that if F is an algebraically closed field of characteristic 3 then $F{\uparrow}_{\mathrm{P\Gamma L}(2,8)}^{S_9}$ does not have a Specht filtration. This gives an interesting example of a permutation module in odd prime characteristic without a Specht filtration. Using more sophisticated techniques, Mikaelian has constructed a family of examples of such modules for fields of characteristic $p > 3$.[2] The existence of such modules is a clear indication that results such as Theorem 3 cannot be obtained by any routine 'reduction mod p' argument.

A preliminary investigation has shown that the situation in characteristic 2 is still more complicated. This is to be expected on theoretical grounds: see [7] and [8] for an introduction to the general theory of Specht filtrations. To demonstrate the difficulties of working in characteristic 2 we end §4 by showing that although the module $\mathbf{F}_2{\uparrow}_{S_3 \wr S_2}^{S_6}$ has a Specht filtration, it *does not* have a Specht filtration with the Specht factors indicated by its ordinary character.

[2]Personal communication, A. Mikaelian, Oxford, July 2007.

1. Proof of Theorem 1

We first classify the multiplicity-free permutation characters given by the actions of symmetric groups on their conjugacy classes. For this we shall need the following lemma, ultimately due to Frobenius, which implies that multiplicity-free permutation characters only come from permutation actions with relatively high degrees of homogeneity.

Lemma 4. *Let $G \leq S_n$ be a permutation group acting on $\{1, 2, \ldots, n\}$. Let π be the permutation character of the action of S_n on the cosets of G. Let $t_r(G)$ be the number of orbits of G on r-subsets of $\{1, 2, \ldots, n\}$. If $0 \leq r \leq n/2$ then*

$$\left\langle \pi, \chi^{(n-r,r)} \right\rangle = \begin{cases} t_r(G) - t_{r-1}(G) & \text{if } r \geq 1 \\ 1 & \text{if } r = 0 \end{cases}. \qquad \square$$

We shall also need the forms of Young's rule and Pieri's rule given in the proposition below. Note that Pieri's rule follows from Young's rule if we conjugate by the sign character, so there is no need for us to use the Littlewood–Richardson rule. (For a proof of Young's rule see [12, Chapter 17]. The modular version of Young's rule proved by James in this reference will be useful to us later—see Theorem 11 in §3 below.)

Proposition 5. *Let $n \geq k \geq 1$ and let μ be a partition of k.*

(i) Young's rule: $(\chi^\mu \times 1_{S_{n-k}})\uparrow_{S_k \times S_{n-k}}^{S_n} = \sum \chi^\lambda$ *where the sum is over all partitions λ obtained from μ by adding $n-k$ nodes, no two in the same column.*

(ii) Pieri's rule: $(\chi^\mu \times \mathrm{sgn}_{S_{n-k}})\uparrow_{S_k \times S_{n-k}}^{S_n} = \sum \chi^\lambda$ *where the sum is over all partitions λ obtained from μ by adding $n-k$ nodes, no two in the same row.* $\square$

Proposition 6. *Let $n \geq 7$, let $x \in S_n$ be a non-identity permutation and let π be the permutation character of S_n acting on the conjugacy class of x. Then π is multiplicity-free if and only if x has one of the cycle types:*

(i) $(2, 1^{n-2})$,

(ii) $(3, 1^{n-3})$,

(iii) (2^m) *when $n = 2m$ or $n = 2m + 1$.*

Furthermore, if π is not multiplicity-free, then either *π contains $\chi^{(n-2,2)}$ more than once* or *x has cycle type (3^m) where $n = 3m$ or $n = 3m + 1$.*

Proof. That π is multiplicity-free in cases (i) and (ii) follows from Young's rule, while case (iii) is given by the $f = 0$ and $f = 1$ cases of the theorem of Inglis, Richardson and Saxl. (As Saxl notes in [15], the $f = 0$ case of this theorem dates back at least to Thrall: see [16, Theorem III].)

Now suppose that π is multiplicity-free. Applying Lemma 4 with the character $\chi^{(n-1,1)}$ shows that $t_1(C_{S_n}(x)) \leq 2$, and hence $C_{S_n}(x)$ has either 1 or 2 orbits on $\{1,\ldots,n\}$. Similarly, applying Lemma 4 with the character $\chi^{(n-2,2)}$ shows that

$$t_2\big(C_{S_n}(x)\big) - t_1\big(C_{S_n}(x)\big) \leq 1, \tag{1}$$

and hence $C_{S_n}(x)$ has at most 3 orbits on the 2-subsets of $\{1,\ldots,n\}$.

Suppose first of all that $C_{S_n}(x)$ is transitive on $\{1,\ldots,n\}$. Then x must have cycle type (l^m) for some $l \geq 2$ and $m \geq 1$ such that $n = lm$. The centralizer $C_{S_n}(x)$ is permutation isomorphic to the wreath product $C_l \wr S_m \leq S_n$. It is not hard to see that the number of orbits of $C_l \wr S_m$ on unordered pairs from $\{1,\ldots,n\}$ is

$$\begin{cases} \lfloor l/2 \rfloor + 1 & \text{if } m > 1 \\ \lfloor l/2 \rfloor & \text{if } m = 1 \end{cases}.$$

Comparing with (1), this shows that if π is multiplicity-free then $l \leq 3$.

Now suppose that $C_{S_n}(x)$ has 2 orbits on $\{1,\ldots,n\}$. The previous paragraph counts the number of orbits of $C_{S_n}(x)$ on unordered pairs with both elements lying in a single orbit of $C_{S_n}(x)$ on $\{1,\ldots,n\}$. It is clear that there is exactly one orbit involving unordered pairs of the form $\{i,j\}$ with i and j taken from different orbits of $C_{S_n}(x)$. We leave it to the reader to check that these remarks imply that either $n = 2m+1$ and x has cycle type $(2^m,1)$, or $n = 3m+1$ and x has cycle type $(3^m,1)$.

To finish the proof we must show that if x has cycle type (3^m) or $(3^m,1)$ then π is not multiplicity-free, even though it contains $\chi^{(n-2,2)}$ only once. The simplest way to do this seems to be to count degrees. Let t_n be the sum of the degrees of all the irreducible characters of S_n. We shall show that $\pi(1) > t_n$ whenever $n \geq 12$. This leaves only three cases to be analysed separately.

It follows from the theorem of Inglis, Richardson and Saxl that t_n is the number of elements of S_n of order at most 2 (of course this result can also be seen in other ways, for example via the Frobenius–Schur count of involutions, or the Robinson–Schensted correspondence). From this it follows that $t_n = t_{n-1} + (n-1)t_{n-2}$ for $n \geq 2$ and hence that $2t_{n-1} \leq t_n \leq nt_{n-1}$ for $n \geq 2$. These results imply that

$$t_{3(m+1)} = (6m+4)t_{3m} + 9m(m+1)t_{3m-1} \leq \tfrac{1}{2}(9m^2 + 21m + 8)t_{3m}.$$

Let $u_n = n!/|C_{S_n}(x)|$ be the degree of π. A short inductive argument using the last inequality shows that $t_{3m} < u_{3m}$ for all $m \geq 4$. Now, provided that $m \geq 4$, we have

$$t_{3m+1} \leq (3m+1)t_{3m} < (3m+1)u_{3m} = u_{3m+1}$$

which is the other inequality we require.

When $n = 10$, one finds that $\pi(1) = 22400$ and $t_{10} = 9496$, and so the degree-counting approach also works in this case. The remaining two cases can be checked by hand; one source for the required character tables is [11, Appendix I.A]. One finds that if x has cycle type $(3,3,1)$ then π contains $\chi^{(3,1^4)}$ twice, while if x has cycle type $(3,3,3)$ then π contains both $\chi^{(5,2,2)}$ and $\chi^{(4,2,1,1,1)}$ twice. $\square$

For $n \leq 6$, one can show by direct calculation that if the permutation character of S_n acting on the conjugacy class of a non-identity element x is multiplicity-free, then x has one of the cycle types in the table below. Note that if $n \leq 4$ then all non-identity classes appear.

n	cycle types
2	(2)
3	$(2,1)$, (3)
4	$(2,1^2)$, $(2,2)$, $(3,1)$, (4)
5	$(2,1^3)$, $(2,2,1)$, $(3,1^2)$, $(3,2)$
6	$(2,1^4)$, (2^3), $(3,1^3)$, $(3,3)$

FIGURE 1. Non-identity conjugacy classes of symmetric groups of degree ≤ 6 whose associated permutation character is multiplicity-free.

We are now ready to prove Theorem 1. Let $n \geq 7$ and let $k \leq n$. Let $x \in S_k$ be a fixed-point-free permutation, let θ be a 1-dimensional character of S_{n-k}, and let $\psi = (1_{C_{S_k}(x)} \times \theta)\big\uparrow_{C_{S_k}(x) \times S_{n-k}}^{S_n}$. If θ is the trivial character then ψ is merely the permutation character of S_n acting on the conjugacy class of S_n containing x, so the result follows from Proposition 6.

We may therefore assume that $k < n$ and that $\theta = \mathrm{sgn}_{S_{n-k}}$. Since

$$\psi = (1_{C_{S_k}(x)}\big\uparrow^{S_k} \times \theta)\big\uparrow_{S_k \times S_{n-k}}^{S_k},$$

if ψ is multiplicity-free, then $1_{C_{S_k}(x)}\uparrow^{S_k}$ must also be multiplicity-free. If $C_{S_k(x)}$ is not transitive on $\{1,\ldots,k\}$ then we have seen that

$$\left\langle 1_{C_{S_k}(x)}\big\uparrow^{S_k}, \chi^{(k-1,1)} \right\rangle \geq 1.$$

It now follows from Pieri's rule that ψ contains $(k, 1^{n-k})$ at least twice. Hence, $C_{S_n}(x)$ acts transitively, and by Proposition 6 and the table above, either x is a fixed-point-free involution in S_k, or x has cycle type (3), (4) or (3^2) with $k = 3$, 4 or 6 respectively.

If x is a fixed-point-free involution then the theorem of Inglis, Richardson and Saxl states that ψ is multiplicity-free. If x is a 3-cycle then it follows from Pieri's rule that π is multiplicity-free. If x is a 4-cycle then

$$\psi = \big((\chi^{(4)} + \chi^{(2,2)} + \chi^{(2,1,1)}) \times \mathrm{sgn}_{S_{n-4}}\big)\big\uparrow_{S_4 \times S_{n-4}}^{S_n},$$

which contains $\chi^{(2,2,1^{n-4})}$ twice. Similarly, if x has cycle type (3^2) then

$$\psi = \big((\chi^{(6)} + \chi^{(4,2)} + \chi^{(4,1,1)} + \chi^{(3,1^3)} + \chi^{(2,2,2)} + \chi^{(2,1^4)}) \times \mathrm{sgn}_{S_{n-6}}\big)\big\uparrow_{S_6 \times S_{n-6}}^{S_n},$$

which contains $\chi^{(4,1^{n-4})}$ twice. This completes the proof of Theorem 1.

2. Proof of Theorem 2

A very large step towards classifying the multiplicity-free permutation characters of symmetric groups was made by Saxl in [15]. In this paper Saxl gives a list of subgroups of S_n for $n \geq 19$, which he proves contains all subgroups G such that the permutation character of S_n acting on the cosets of G is multiplicity-free. Our contribution is to prune his list of the unwanted subgroups. There are several interesting features that still remain for us to discover, and to obtain the most uniform result, we must assume that $n \geq 66$.

Since we shall frequently need to refer to it, we give a verbatim statement of Saxl's theorem from [15, page 340]. There is a minor error in case (v), to which the groups $A_n \cap (S_{n-k} \times G_k)$ for $k = 5, 6$ should be added. (It follows from the argument at the bottom of page 342 of Saxl's paper that these groups should be considered for inclusion, and in fact both give rise to multiplicity-free characters.)

Theorem (Saxl)**.** *Let S_n be multiplicity-free on the set of cosets* [denoted Ω] *of a subgroup G. Assume that $n > 18$. Then one of:*

(i) $A_{n-k} \times A_k \leq G \leq S_{n-k} \times S_k$ *for some k with* $0 \leq k < n/2$*;*

(ii) $n = 2k$ *and* $A_k \times A_k < G \leq S_k \wr S_2$*;*

(iii) $n = 2k$ *and* $G \leq S_2 \wr S_k$ *of index at most* 4*;*

(iv) $n = 2k + 1$ *and G fixes a point of Ω and is one of the groups in* (ii) *or* (iii) *on the rest of* Ω*; or*

(v) $A_{n-k} \times G_k \leq G \leq S_{n-k} \times G_k$ *where k is* 5, 6 *or* 9 *and G_k is Frobenius of order* 20, $\mathrm{PGL}(2,5)$ *or* $\mathrm{P\Gamma L}(2,8)$ *respectively.*

We now examine each of Saxl's cases in turn. The most interesting case (iii) is left to the end, and we consider case (iv) together with (ii) and (iii). We shall frequently need the well-known result (see for example [12, 6.6]) that if λ is a partition then

$$\chi^{\lambda} \times \mathrm{sgn} = \chi^{\lambda'} \qquad (2)$$

where λ' is the conjugate partition to λ. (Recall that λ' is the partition defined by $\lambda'_i = |\{j : \lambda_j \geq i\}|$; the diagram of λ' is obtained from the diagram of λ by reflecting it in its main diagonal.) We shall also frequently use the fact that if $H < G < S_n$ and $1_H{\uparrow}^{S_n}$ is multiplicity-free, then $1_G{\uparrow}^{S_n}$ is also multiplicity-free.

2.1. **Case (i).** If $k = 0$ or $k = 1$ then the subgroups from this case clearly give multiplicity-free characters. They contribute to our case (a1). If $2 \leq k \leq n/2$ then it follows from (2) together with Young's rule and Pieri's rule that

$$\begin{aligned} 1_{A_{n-k}\times A_k}{\uparrow}^{S_n} &= (1_{S_{n-k}\times S_k}{\uparrow}^{S_n} + \mathrm{sgn}_{S_{n-k}} \times 1_{S_k}{\uparrow}^{S_n})(1+\mathrm{sgn}_{S_n}) \\ &= \sum_{i=0}^{k} \chi^{(n-i,i)} + \sum_{i=0}^{k} \chi^{(n-i,i)'} + \chi^{(n-k,1^k)} + \chi^{(n-k,1^k)'} \\ &\qquad + \chi^{(n-k+1,1^{k-1})} + \chi^{(n-k+1,1^{k-1})'}. \end{aligned} \tag{3}$$

Hence, for k in this range, $1_{A_{n-k}\times A_k}{\uparrow}^{S_n}$ is multiplicity-free unless $k = 2$ or $k = \lfloor n/2 \rfloor$. When $k = 2$ it is easily seen that $1_{S_{n-2}}{\uparrow}^{S_n}$ is not multiplicity-free, while if G is one of the other two index 2-subgroups of $S_{n-2} \times S_2$, namely $A_{n-2} \times S_2$ or $A_n \cap (S_{n-2} \times S_2)$, then $1_G{\uparrow}^{S_n}$ *is* multiplicity-free. This gives the remaining groups in our case (a1) and the groups in case (a2).

If n is even then we have already dealt with all the groups from Saxl's case (i). If $n = 2k+1$ is odd then we still have to deal with the subgroups of $S_{k+1} \times S_k$ properly containing $A_{k+1} \times A_k$. A calculation similar to (3) shows that all these groups give multiplicity-free characters; they appear in our case (b2).

2.2. **Case (ii).** There are three index 2 subgroups of $S_k \wr S_2$, namely $S_k \times S_k$, $A_{2k} \cap (S_k \wr S_2)$ and one other, which we shall denote by Γ_k. Figure 2 overleaf shows the lattice of subgroups we must consider; note that they are in bijection with the subgroups of the dihedral group of order 8.

From (3) we know that $1_{A_k \times A_k}{\uparrow}^{S_{2k}}$ is not multiplicity-free. However, it turns out that every subgroup of $S_k \wr S_2$ which properly contains $A_k \times A_k$ does give a multiplicity-free permutation character. These groups appear in our case (b1). For later use we give their permutation characters in full. We shall need the character ψ_k of $S_k \wr S_2$ defined by the composition of maps $S_k \wr S_2 \twoheadrightarrow S_2 \cong \{\pm 1\}$; note that $\Gamma_k = \ker \psi_k \, \mathrm{sgn}_{S_k \wr S_2}$. Example 2.3 in [15] tells us that

$$1_{S_k \wr S_2}{\uparrow}^{S_{2k}} = \sum_{i=0}^{\lfloor k/2 \rfloor} \chi^{(2k-2i,2i)}. \tag{4}$$

Given (4), it follows from the known decomposition of $1_{S_k \times S_k}{\uparrow}^{S_{2k}}$ that

$$\psi_k{\uparrow}_{S_k \wr S_2}^{S_{2k}} = \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \chi^{(2k-2i-1,2i+1)}. \tag{5}$$

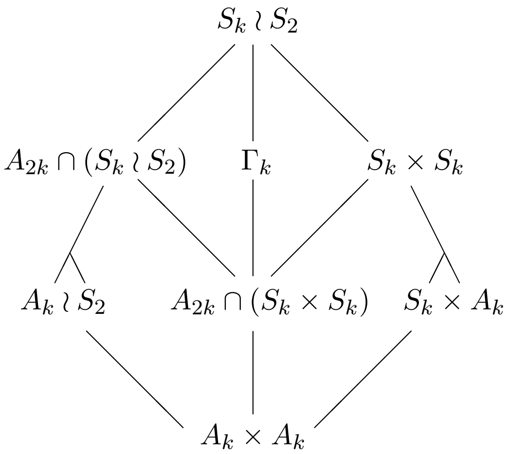

FIGURE 2. Subgroups of $S_k \wr S_2$ containing $A_k \times A_k$ when k is even. If k is odd then the labels $A_{2k} \cap (S_k \wr S_2)$ and Γ_k should be swapped. The forked lines to the subgroups $S_k \times A_k$ and $A_k \wr S_2$ indicate that they appear in two conjugate copies.

Using (2) and (5) together with Young's rule and Pieri's rule we find that

$$\begin{aligned}
1_{A_{2k} \cap (S_k \wr S_2)} \big\uparrow^{S_{2k}} &= 1_{S_k \wr S_2} \big\uparrow^{S_{2k}} + \operatorname{sgn}_{S_k \wr S_2} \big\uparrow^{S_{2k}} \\
&= \sum_{i=0}^{\lfloor k/2 \rfloor} \chi^{(2k-2i,2i)} + \sum_{i=0}^{\lfloor k/2 \rfloor} \chi^{(2k-2i,2i)'}, \\
1_{\Gamma_k} \big\uparrow^{S_{2k}} &= 1_{S_k \wr S_2} \big\uparrow^{S_{2k}} + \psi_k \operatorname{sgn}_{S_k \wr S_2} \big\uparrow^{S_{2k}} \\
&= \sum_{i=0}^{\lfloor k/2 \rfloor} \chi^{(2k-2i,2i)} + \sum_{i=0}^{\lfloor (k-1)/2 \rfloor} \chi^{(2k-2i-1,2i+1)'}.
\end{aligned}$$

Similar calculations give the permutation characters induced from the index 4 subgroups:

$$\begin{aligned}
1_{S_k \times A_k} \big\uparrow^{S_{2k}} &= \sum_{i=0}^{k} \chi^{(2k-i,i)} + \chi^{(k+1,1^{k-1})} + \chi^{(k,1^k)}, \\
1_{A_{2k} \cap (S_k \times S_k)} \big\uparrow^{S_{2k}} &= \sum_{i=0}^{k} \chi^{(2k-i,i)} + \sum_{i=0}^{k} \chi^{(2k-i,i)'}, \\
1_{A_k \wr S_2} \big\uparrow^{S_{2k}} &= \sum_{i=0}^{\lfloor k/2 \rfloor} \chi^{(2k-2i,2i)} + \chi^{(k+1,1^{k-1})} + \chi^{(k,1^k)} + \alpha_k,
\end{aligned}$$

where in the last line

$$\alpha_k = \begin{cases} \sum_{i=0}^{k/2} \chi^{(2k-2i,2i)'} & \text{if } k \text{ is even} \\ \sum_{i=0}^{(k-1)/2} \chi^{(2k-2i-1,2i+1)'} & \text{if } k \text{ is odd.} \end{cases}$$

To decide which of these characters remain multiplicity-free when induced from S_{2k} to S_{2k+1}, and so should be taken from Saxl's case (iv), we first note that

$$1_{S_k \times S_k} \big\uparrow^{S_{2k+1}} = \left(\sum_{r=0}^{k} \chi^{(2k-r,r)} \right) \big\uparrow^{S_{2k+1}}$$

contains $\chi^{(2k,1)}$ twice. (In the second induction above, Young's rule may be replaced with the ordinary branching rule: see [12, Chapter 9].) Hence, if G is any subgroup of $S_k \times S_k$, then $1_G \uparrow^{S_{2k+1}}$ is not multiplicity-free. Similarly one shows that the permutation character induced from $A_k \wr S_2$ is not multiplicity-free, while the characters induced from $A_{2k} \cap (S_k \wr S_2)$ and Γ_k are. This gives the remaining groups in our case (b2).

2.3. **Case (v).** We now turn to Saxl's case (v). The subgroups G_k for $k = 5, 6, 9$ are each $\lfloor k/2 \rfloor$-homogeneous. (It follows from Young's rule and Lemma 4 that this is a necessary condition for the induced characters $1_{G_k \times S_{n-k}} \uparrow^{S_n}$ to be multiplicity-free for every n.) They are: the 2-transitive Frobenius group $G_5 = \langle (12345), (2354) \rangle \leq S_5$; the 3-transitive subgroup $G_6 = \mathrm{PGL}(2,5) \leq S_6$; and the 3-transitive but 4-homogeneous subgroup $G_9 = \mathrm{P\Gamma L}(2,8) \leq S_9$. Here $\mathrm{P\Gamma L}(2,8)$ denotes the split extension of $\mathrm{PGL}(2,8)$ given by the order 3 Frobenius twist $F : \mathbf{F}_8 \to \mathbf{F}_8$.

Calculation using Young's rule shows that the permutation characters

$$\begin{aligned} 1_{G_5} \times 1_{S_{n-5}} \big\uparrow^{S_n} &= \left(\chi^{(5)} + \chi^{(2,2,1)}\right) \times 1_{S_{n-5}} \big\uparrow^{S_n}_{S_5 \times S_{n-5}}, \\ 1_{G_6} \times 1_{S_{n-6}} \big\uparrow^{S_n} &= \left(\chi^{(6)} + \chi^{(2,2,2)}\right) \times 1_{S_{n-6}} \big\uparrow^{S_n}_{S_6 \times S_{n-6}} \end{aligned}$$

are always multiplicity-free. A nice way to obtain these equations uses the outer automorphism of S_6: if $H \cong S_5$ is a point stabiliser in S_6, then H is mapped under an outer automorphism of S_6 to a subgroup permutation isomorphic to $G_6 = \mathrm{PGL}(2,5)$. Inspection of the character table of S_6 shows that the constituent $\chi^{(5,1)}$ of $1_H \uparrow^{S_6}$ is mapped to the constituent $\chi^{(2,2,2)}$ of $1_{G_5} \uparrow^{S_6}$. Since $G_6 \cap H$ is conjugate in S_5 to G_5, the character induced from G_5 can then be obtained by restriction.

The remaining character from Saxl's case (v) is

$$1_{G_9} \times 1_{S_{n-9}} \big\uparrow^{S_n} = \left(\chi^{(9)} + \chi^{(1^9)} + \chi^{(5,1^4)} + \chi^{(4,4,1)} + \chi^{(3,2^3)}\right) \times 1_{S_{n-9}} \big\uparrow^{S_n}_{S_9 \times S_{n-9}}, \quad (6)$$

which is always multiplicity-free. We outline one way to obtain this equation. One easily checks that $\mathrm{P\Gamma L}(2,8) \leq A_9$, so by (2), χ^λ appears in $\pi = 1_{\mathrm{P\Gamma L}(2,8)} \uparrow^{S_9}$ if and only if $\chi^{\lambda'}$ appears. By Lemma 4, none of $\chi^{(8,1)}, \chi^{(7,2)}, \chi^{(6,3)}, \chi^{(5,4)}$, or their conjugates, appears in π. From the equation

$$1_{S_{n-r}} \times \mathrm{sgn}_{S_r} \big\uparrow^{S_n} = \chi^{(n-r,1^r)} + \chi^{(n-r+1,1^{r-1})} \quad \text{if } 1 \leq r < n$$

and Frobenius reciprocity one sees that

$$\left\langle \pi, \chi^{(9-r,1^r)} \right\rangle = \rho_r - \rho_{r-1} + \cdots + (-1)^r \rho_0 \quad \text{if } 0 \leq r < 9,$$

where $\rho_i = 1$ if $\mathrm{P\Gamma L}(2,8) \cap (S_{9-r} \times S_r) \leq S_{9-r} \times A_r$, and $\rho_i = 0$ otherwise. (Since $\mathrm{P\Gamma L}(2,8)$ is 4-homogeneous, it does not matter which subgroup $S_{9-r} \times S_r \leq S_9$ we choose.) Clearly $\rho_0 = \rho_1 = 1$. The group $\mathrm{PGL}(2,8)$ has a unique conjugacy class of elements of even order; these are involutions, and since $\mathrm{PGL}(2,8)$ is sharply 3-transitive, they must act with cycle type $(2^4, 1)$. Hence $\rho_2 = \rho_3 = 0$. It follows from the identity $(Fg)^3 = g^{F^2} g^F g$ for $g \in \mathrm{PGL}(2,8)$, that the only new even order that appears when we extend $\mathrm{PGL}(2,8)$ to $\mathrm{P\Gamma L}(2,8)$ is 6. Therefore no 4-cycles appear in the cycle decomposition of any element of $\mathrm{P\Gamma L}(2,8)$, and $\rho_4 = 1$. Hence, apart from $\chi^{(9)}$ and $\chi^{(1^9)}$, the only hook character to appear in π is $\chi^{(5,1^4)}$. We now have

$$\pi = \chi^{(9)} + \chi^{(1^9)} + \chi^{(5,1^4)} + \psi$$

where ψ has degree 168. With the exception of $\chi^{(3,3,3)}$ (which has degree 42) and the pair $\chi^{(4,4,1)}$, $\chi^{(3,2^3)}$ (each of degree 84), all the irreducible characters of S_9 that are still eligible to appear in ψ have too high a degree. If $\chi^{(3,3,3)}$ appears four times, then we would have $\pi\big((12345)\big) = 10$; the required character values may be computed by hand, or found in [11, Appendix I.A]. However, $\mathrm{P\Gamma L}(2,8)$ contains no elements of order 5, so clearly $\pi((12345)) = 0$. Equation (6) follows.

It is straightforward to check using the formulae

$$1_{G_k} \times 1_{A_{n-k}} \big\uparrow^{S_n} = \big(1_{G_k} \times 1_{S_{n-k}}\big) \big\uparrow^{S_n} + \Big(1_{G_k} \times \mathrm{sgn}_{S_{n-k}}\Big) \big\uparrow^{S_n},$$
$$1_{A_n \cap (G_k \times S_{n-k})} \big\uparrow^{S_n} = \big(1_{G_k} \times 1_{S_{n-k}}\big) \big\uparrow^{S_n} + \big(1_{G_k} \times 1_{S_{n-k}}\big) \big\uparrow^{S_n} \times \mathrm{sgn}_{S_n}$$

and Pieri's rule that, provided $n \geq 20$, the characters $1_{G_k} \times 1_{A_{n-k}} \uparrow^{S_n}$ and $1_{A_n \cap (G_k \times S_{n-k})} \uparrow^{S_n}$ are also multiplicity-free. This gives the groups in our case (e).

2.4. **Case (iii).** It remains to deal with Saxl's case (iii): subgroups of $S_2 \wr S_k$ of index at most 4. By the theorem of Inglis, Richardson and Saxl, $1_{S_2 \wr S_k} \uparrow^{S_{2k}}$ is multiplicity-free. Moreover, this character is still multiplicity-free if we induce up to S_{2k+1}, since

$$1_{S_2 \wr S_k} \big\uparrow^{S_{2k+1}} = \sum_{\lambda} \chi^{\lambda}$$

where the sum is over all partitions λ of $2k+1$ with exactly one odd part. We therefore take $S_2 \wr S_k$ from Saxl's case (iv). This gives our case (c). It now only remains to look at the proper subgroups of $S_2 \wr S_k$.

Let H be the unique normal subgroup of $S_2 \wr S_k$ of index 2 in the base group $S_2 \times \cdots \times S_2$. A straightforward argument shows that, provided $k \geq 5$, the group $H \rtimes A_k$ is the only subgroup of $S_2 \wr S_k$ of index 4. This subgroup is normal in $S_2 \wr S_k$, and the quotient group $S_2 \wr S_k \big/ H \rtimes A_k$ is isomorphic to $C_2 \times C_2$. It follows that there are three subgroups of index 2 in $S_2 \wr S_k$, namely $H \rtimes S_k$, $S_2 \wr A_k$, and one other, which we shall denote by Δ_k. The subgroup lattice is shown in Figure 3 below.

$$
\begin{tikzcd}
 & S_2 \wr S_k \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
H \rtimes S_k \arrow[dr, dash] & \Delta_k \arrow[d, dash] & S_2 \wr A_k \arrow[dl, dash] \\
 & H \rtimes A_k &
\end{tikzcd}
$$

FIGURE 3. Subgroups of index at most 4 in $S_2 \wr S_k$

It is easy to check that the subgroup $H \rtimes S_k$ is equal to $A_{2k} \cap (S_2 \wr S_k)$. Hence

$$1_{H \rtimes S_k}\big\uparrow^{S_{2k}} = 1_{S_2 \wr S_k}\big\uparrow^{S_{2k}} + \mathrm{sgn}_{S_2 \wr S_k}\big\uparrow^{S_{2k}}$$

and so

$$1_{H \rtimes S_k}\big\uparrow^{S_{2k}} = \sum \chi^{\lambda} + \sum \chi^{\lambda'} \tag{7}$$

where the sums are over all partitions λ of $2k$ with only even parts.

From now on, we shall say that a partition all of whose parts are even is *even*. We see from (7) that $1_{H \rtimes S_k}\uparrow^{S_{2k}}$ fails to be multiplicity-free if and only if there is an even partition λ whose conjugate λ' is also even. If k is even then (k, k) is such a partition, while if k is odd then it is clear that no such partition can exist. This gives case (d) of Theorem 2.

Suppose that k is odd. If we induce the character $1_{H \rtimes S_k}\uparrow^{S_{2k}}$ up to S_{2k+1}, then we obtain the constituent $\chi^{(k+1,k)}$ twice: once by adding a node to the even partition $(k+1, k-1)$, and once by adding a node to the partition (k, k), whose conjugate (2^n) is even. The group $H \rtimes S_k$ is therefore not included in those coming from Saxl's case (iv).

To complete the proof of Theorem 2, it suffices to show that if $k \geq 33$, then neither of the permutation characters induced from the other two index 2 subgroups of $S_2 \wr S_k$ is multiplicity-free. In order to describe the constituents of these permutation characters we shall use the following notation: if $\alpha = (a_1, a_2, \ldots, a_r)$ is a partition of k with distinct parts, let $2[\alpha] = 2[a_1, \ldots, a_r]$ denote the partition λ of $2k$ whose leading diagonal hook lengths are $2a_1, \ldots, 2a_r$, and such that $\lambda_i = a_i + i$ for $1 \leq i \leq r$. For instance, Figure 4 overleaf shows $2[4, 3, 1]$.

We can now state the following lemma, which is the analogue of (5) in §2.2.

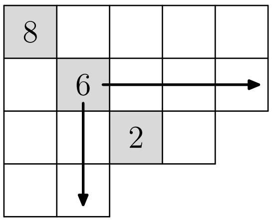

FIGURE 4. Leading diagonal hook lengths in $2[4,3,1] = (5,5,4,2)$

Lemma 7. *Let $k \geq 2$ and let θ_k be the character of $S_2 \wr S_k$ defined by the composition of maps $S_2 \wr S_k \twoheadrightarrow S_k \overset{\mathrm{sgn}}{\twoheadrightarrow} \{\pm 1\}$. Then $1_{S_2 \wr A_k}\big\uparrow^{S_2 \wr S_k} = 1_{S_2 \wr S_k} + \theta_k$ and*

$$\theta_k \big\uparrow^{S_{2k}} = \sum \chi^{2[\alpha]}$$

where the sum is over all partitions α of k with distinct parts.

Before proving Lemma 7 we use it to complete the proof of Theorem 2. By the first statement in the lemma we have

$$1_{S_2 \wr A_k}\big\uparrow^{S_{2k}} = 1_{S_2 \wr S_k}\big\uparrow^{S_{2k}} + \theta_k \big\uparrow^{S_{2k}}.$$

Hence, $1_{S_2 \wr A_k}\big\uparrow^{S_{2k}}$ fails to be multiplicity-free if and only if there is an even partition of the form $2[\alpha]$. Now, the partition $2[a_1, \ldots, a_r]$ is even if and only if the following conditions hold: a_{2i-1} is odd and $a_{2i} = a_{2i-1} - 1$ for all $i \leq r/2$, and, if r is odd, then $a_r = 1$. It follows, on setting $2b_i = a_{2i-1} - 1$, that $1_{S_2 \wr A_k}\big\uparrow^{S_{2k}}$ fails to be multiplicity-free if and only if there is a strictly decreasing sequence of positive integers $(b_1, \ldots, b_s)$ such that either

$$k = 4\sum_{i=1}^{s} b_i + s \quad \text{or} \quad k = 4\sum_{i=1}^{s} b_i + s + 1.$$

One now shows, by looking at the possible values of k mod 4, that provided $k \geq 25$, at least one of these equations has a solution. For example, if $k = 4l$ with $l \geq 7$, then one can solve the second equation by taking $s = 3$, $b_1 = l - 4$, $b_2 = 2$ and $b_3 = 1$. The bound on k is strict: when $k = 24$, neither equation is soluble, and hence the permutation character $1_{S_2 \wr A_{24}}\big\uparrow^{S_{48}}$ is multiplicity-free.

Finally we consider the subgroup Δ_k. It is easy to check that

$$1_{\Delta_k}\big\uparrow^{S_2 \wr S_k} = 1_{S_2 \wr S_k} + \theta_k \,\mathrm{sgn}_{S_2 \wr S_k}.$$

Hence

$$1_{\Delta_k}\big\uparrow^{S_{2k}} = 1_{S_2 \wr S_k}\big\uparrow^{S_{2k}} + \theta_k\big\uparrow^{S_{2k}} \times \mathrm{sgn}_{S_{2k}}.$$

By (2) and Lemma 7, we see that $1_{\Delta_k}\big\uparrow^{S_{2k}}$ fails to be multiplicity-free if and only there is a partition $2[\alpha]$ whose *conjugate* is even. The partition $2[a_1, \ldots, a_r]$ has an even conjugate if and only if a_{2i-1} is even and $a_{2i} =$

$a_{2i-1} - 1$ for all $i \leq r/2$, and r is even. It follows, on setting $2b_i = a_{2i-1}$, that $1_{\Delta_k}\!\uparrow^{S_{2k}}$ fails to be multiplicity-free if and only if there is a strictly decreasing sequence of positive integers $(b_1, \ldots, b_s)$ such that

$$k = 4\sum_{i=1}^{s} b_i - s.$$

By a very similar argument to before, we now find that $1_{\Delta_k}\!\uparrow^{S_{2k}}$ is not multiplicity-free if $k \geq 33$. Again this bound is strict.

Proof of Lemma 7. It is easy to see that θ_k is the unique non-trivial constituent of $1_{S_2 \wr A_k}\!\uparrow^{S_2 \wr S_k}$. To proceed further, we adapt the proof of the decomposition of $1_{S_2 \wr S_k}\!\uparrow^{S_{2k}}$ attributed to James and Saxl in [15, Example 2.2]. Given a partition λ, we define the *rank* of λ to be the maximum integer r such that $\lambda_r \geq r$. (Thus the partition $2[a_1, \ldots, a_r]$ has rank r.) Let $\phi_k = \theta_k\!\uparrow^{S_{2k}}$. To prove the lemma, we must show that $\phi_k = \sum \chi^{2[\alpha]}$, where the sum is over all partitions α of k with distinct parts.

By an easy application of Mackey's lemma (see [1, Theorem 3.3.4]) we have

$$\phi_k\big\downarrow_{S_{2k-1}} = \phi_{k-1}\big\uparrow^{S_{2k-1}}.$$

It follows by induction that

$$\phi_k\big\downarrow_{S_{2k-1}} = \sum \chi^{2[\alpha]}\big\uparrow^{S_{2k-1}} \tag{8}$$

where the sum is over all partitions α of $k-1$ with distinct parts. We now calculate, again using Mackey's lemma, that

$$\begin{aligned}\left\langle 1_{C_2 \wr A_k}\big\uparrow^{S_{2k}}, 1_{S_k} \times \mathrm{sgn}_{S_k}\big\uparrow^{S_{2k}}_{S_k \times S_k}\right\rangle &= \left\langle 1_{C_2 \wr A_k}\big\uparrow^{S_{2k}}\big\downarrow_{S_k \times S_k}, 1_{S_k} \times \mathrm{sgn}_{S_k}\right\rangle_{S_k \times S_k} \\ &= \sum_g \left\langle 1, 1_{S_k} \times \mathrm{sgn}_{S_k}\right\rangle_{(C_2 \wr A_k)^g \cap (S_k \times S_k)} \\ &= \sum_g \left\{ \begin{array}{l} 1 \ : \ (C_2 \wr A_k)^g \cap (S_k \times S_k) \leq S_k \times A_k \\ 0 \ : \ \text{otherwise} \end{array} \right. \\ &\geq 1,\end{aligned}$$

where in the sums g runs over a set of representatives for the double cosets of $C_2 \wr A_k$ and $S_k \times S_k$ in S_{2k}. It follows from Pieri's rule that ϕ_k contains either $\chi^{(k+1, 1^{k-1})}$ or $\chi^{(k, 1^k)}$ with positive multiplicity. From (8) we see that the latter character cannot occur in ϕ_k, while $\chi^{(k+1, 1^{k-1})}$ can occur at most once. Thus ϕ_k contains $\chi^{2[k]}$ exactly once.

Suppose now that χ^λ is a constituent of ϕ_k. If λ has rank 3 or more, it follows immediately from (8) that $\lambda = 2[\alpha]$ for some α. The rank 1 and rank 2 possibilities need a little more care, but in close analogy with Saxl's argument, one can rule out the appearance of any unwanted characters by using the known occurrence of $\chi^{2[k]}$. Finally, suppose that $\alpha = (a_1, \ldots, a_r)$

is a partition of k with distinct parts and that $\chi^{2[\alpha]}$ does not appear in ϕ_k. Then, if μ is the partition obtained from $2[\alpha_1, \ldots, \alpha_r]$ by removing a node from row r, χ^μ does not appear in $\phi_k\downarrow_{S_{2k-1}}$, in contradiction to (8). □

2.5. **Corollaries.** Working through the cases in Theorem 2 we get a complete list of all the irreducible characters of symmetric groups that can be obtained as a constituent of a multiplicity-free permutation representation.

Corollary 8. *Let $n \geq 66$ and let λ be a partition of n. The irreducible character χ^λ is a constituent of a multiplicity-free permutation character of S_n if and only if (at least) one of:*

(1) *λ has at most two rows or at most two columns;*
(2) *$\lambda = (n-i, 1^i)$ for some i with $0 \leq k < n$;*
(3) *$\lambda = (2k-i, i, 1)'$ where $n = 2k$ and $1 \leq i \leq k$;*
(4) *λ has at most one row of odd length;*
(5) *λ has columns all of even length and $n \equiv 2 \mod 4$;*
(6) *λ can be obtained by adding nodes to one of the following partitions*

$$(2,2,1), (2,2,2), (5,1^4), (4,4,1), (3,2,2,2)$$

subject to the restriction that all added nodes are in different columns;

(7) *λ can be obtained by adding nodes to one of the following partitions*

$$(3,2), (3,3), (2,2,1), (2,2,2), (5,1^4), (4,4,1), (3,2,2,2)$$

subject to the restriction that all added nodes are in different rows. □

Proof. Cases (1) and (2) give the characters coming from cases (a1), (a2) and (b1) of Theorem 2. If $n = 2k+1$ then it follows from the explicit calculations in §2.2 that the groups in case (b2) contribute the further characters with labels $(2k-2i, 2i, 1)$, $(2k-2i, 2i, 1)'$ for $1 \leq i \leq k/2$ and $(2k-2i-1, 2i+1, 1)'$ for $0 \leq i \leq (k-1)/2$. The first family is subsumed by case (4); the others form case (3). The remaining cases are straightforward: case (4) comes directly from (c), case (5) from (d) and cases (6) and (7) from (e). □

The following immediate corollary of Theorem 2 is also of interest.

Corollary 9. *Let $n \geq 66$. Suppose that G is a subgroup of S_n such that the permutation character of S_n acting on the cosets of G is multiplicity-free. If the permutation character of S_{n+1} acting on the cosets of G is also multiplicity-free then* either $G \geq A_n$ or *$n = 2k$ is even and either $G = S_2 \wr S_k$ or G is a transitive subgroup of $S_k \wr S_2$ of index at most 2.* □

3. Proof of Theorem 3

We begin by collecting the background results we need for the proof of Theorem 3. We shall distinguish between inner tensor products, denoted $\otimes$, and outer tensor products, denoted $\boxtimes$.

Lemma 10. *Let F be a field. If U is a module for FS_n with a Specht filtration then $U \otimes sgn$ has a filtration by the duals of Specht modules. In particular, if M is a self-dual module for FS_n with a Specht filtration, then $M \otimes \mathrm{sgn}$ also has a Specht filtration.*

Proof. By [12, Theorem 8.15], if λ is any partition then

$$S^\lambda \otimes \mathrm{sgn} \cong \left(S^{\lambda'}\right)^\star. \tag{9}$$

(This is the modular version of (2) above.) Since the functor sending an FS_n-module U to $U \otimes \mathrm{sgn}$ is clearly exact, this is all we need to prove the lemma. $\square$

Theorem 11. *Let F be a field and let $n > k \geq 1$. If λ is a partition of k then*

$$S^\lambda \boxtimes F_{S_{n-k}} \big\uparrow_{S_k \times S_{n-k}}^{S_n}$$

has a Specht filtration, with the Specht factors given by Young's rule. Similarly

$$S^\lambda \boxtimes \mathrm{sgn}_{S_{n-k}} \big\uparrow_{S_k \times S_{n-k}}^{S_n}$$

has a Specht filtration, with the Specht factors given by Pieri's rule.

Proof. The first statement follows from James' modular version of Young's rule [12, Corollary 17.14]. The second may be deduced from the first by using (9). $\square$

Since the functor sending an FS_k-module U to $U \boxtimes F_{S_{n-k}}\uparrow^{S_n}$ is exact, it follows from Theorem 11 that if U is an FS_k-module with a Specht filtration then $U \boxtimes F_{S_{n-k}}\uparrow^{S_n}$ also has a Specht filtration, with the Specht factors given by repeated applications of Young's rule. Naturally there is a similar result for $U \boxtimes \mathrm{sgn}_{S_{n-k}}\uparrow^{S_n}$.

It remains to state two results concerning summands of permutation modules. Both of these have a slightly technical flavour, but neither is at all difficult to apply.

Lemma 12. *Let F be a field of prime characteristic p. If G is a subgroup of S_n such that the permutation character $1_G\uparrow^{S_n}$ is multiplicity-free then all the summands of $F\uparrow_G^{S_n}$ are self-dual.*

Proof. For simplicity, we assume that F is the field with p elements. Let U be an indecomposable direct summand of $F\uparrow_G^{S_n}$. Let $\mathbf{Z}_p$ denote the ring of

p-adic integers. Since U is a direct summand of a permutation module, we may lift U to a $\mathbf{Z}_p S_n$-module V such that V is a direct summand of $\mathbf{Z}_p\!\uparrow_G^{S_n}$ and $V \otimes_{\mathbf{Z}_p} F = U$. (See [1, §3.11] for an outline of this lifting process.)

Suppose that U is not self-dual. Then the lifted module V is not self-dual either. Since $\mathbf{Z}_p\!\uparrow_G^{S_n}$ is self-dual, we may find a summand V' of $\mathbf{Z}_p\!\uparrow_G^{S_n}$ such that $V' \cong V^\star$. As V and V' are non-isomorphic, they are distinct summands of $\mathbf{Z}_p\!\uparrow_G^{S_n}$. But V and $V^\star$ have the same ordinary character. This contradicts our assumption that the character $1_G\!\uparrow^{S_n}$ is multiplicity-free. $\square$

This lemma deals with the assertions about duality in Theorem 3. It may also be used to replace the reference to the author's D. Phil thesis [17, Theorem 6.5.1] in the proof of Theorem 4 of [14].

Finally, we shall often be in the position of knowing that a permutation module $F\!\uparrow_G^{S_n}$ has a Specht filtration, and wishing to prove that the same result holds for each of its summands. Since in Theorem 3 we assume that our ground field F is algebraically closed and of characteristic > 3, we may use the homological algebra approach developed by Hemmer and Nakano in [7]. (For an alternative, slightly less technological approach, see the remark attributed to S. Donkin at the end of §1 of [14].)

Proposition 13. *Let F be an algebraically closed field of prime characteristic $p > 3$. Let M be a module for FS_n with a Specht filtration. If U is a direct summand of M then U also has a Specht filtration.*

Proof. This is immediate from [7, Theorem 3.6.1]. $\square$

We are now ready to prove Theorem 3. By the work of Hemmer and Nakano [7], when $p > 3$, the multiplicities of the factors in a Specht filtration are well-defined. Hence it suffices to show that each of the permutation modules in Theorem 3 has a Specht filtration, with the Specht factors given by its ordinary character.

We start with the modules coming from case (a) of Theorem 2. Suppose that $G = A_k \times A_{n-k}$. Since the ground field F has odd characteristic,

$$\begin{aligned} F\big\uparrow_{A_k \times A_{n-k}}^{S_n} &= \big(F_{S_k} \boxtimes F_{S_{n-k}}\big)\big\uparrow^{S_n} \oplus \big(\mathrm{sgn}_{S_k} \boxtimes F_{S_{n-k}}\big)\big\uparrow^{S_n} \\ &\quad \oplus \Big(\big(F_{S_k} \boxtimes F_{S_{n-k}}\big)\big\uparrow^{S_n} \otimes \mathrm{sgn}_{S_n}\Big) \oplus \Big(\big(\mathrm{sgn}_{S_k} \boxtimes F_{S_{n-k}}\big)\big\uparrow^{S_n} \otimes \mathrm{sgn}_{S_n}\Big). \end{aligned}$$

It follows from Lemma 10 and Theorem 11 that each of the four summands has a Specht filtrations. Proposition 13 now guarantees that *any* indecomposable summand of $F\!\uparrow_{A_k \times A_{n-k}}^{S_n}$ has a Specht filtration. This deals with all the subgroups appearing in case (a), and also the subgroups of $S_{k+1} \times S_k$ in case (b2).

Now suppose that $n = 2k$. We first note that

$$F\big\uparrow_{S_k \times S_k}^{S_{2k}} = F\big\uparrow_{S_k \wr S_2}^{S_{2k}} \oplus \psi_k\big\uparrow_{S_k \wr S_2}^{S_{2k}}$$

where ψ_k is the 1-dimensional representation of $S_k \wr S_2$ defined in §2.2. Hence both of the summands have a Specht filtration. (Here it is essential that F has odd characteristic: see §4.2 for an example when F has characteristic 2.) It follows that

$$F\big\uparrow_{A_{2k}\cap(S_k\wr S_2)}^{S_{2k}} = F\big\uparrow_{S_k\wr S_2}^{S_{2k}} \oplus \left(F\big\uparrow_{S_k\wr S_2}^{S_{2k}}\right) \otimes \mathrm{sgn}_{S_{2k}}$$

and

$$F\big\uparrow_{\Gamma_k}^{S_{2k}} = F\big\uparrow_{S_k\wr S_2}^{S_{2k}} \oplus \left(\psi_k\big\uparrow_{S_k\wr S_2}^{S_{2k}}\right) \otimes \mathrm{sgn}_{S_{2k}}$$

both have Specht filtrations. Moreover, if G is a subgroup of $S_k \wr S_2$ of index 4 then $F\uparrow_G^{S_{2k}}$ is in every case a direct summand of $F\uparrow_{A_k\times A_k}^{S_{2k}}$, and so has a Specht filtration. Hence if G is any subgroup of S_{2k} such that $A_k\times A_k < G \leq S_k\wr S_2$, then $F\uparrow_G^{S_{2k}}$ has a Specht filtration. By Theorem 11, $F\uparrow_G^{S_{2k+1}}$ also has a Specht filtration. These remarks deal with all the remaining subgroups in case (b).

In cases (c) and (d), a considerable amount of work is done for us by Theorem 2 of [14], which implies that $F\uparrow_{S_2\wr S_k}^{S_{2k}}$ has a Specht filtration. By Theorem 11, $F\uparrow_{S_2\wr S_k}^{S_{2k+1}}$ also has a Specht filtration. The argument giving (7) in §2.4 shows that

$$F_{H\rtimes S_k}\big\uparrow^{S_{2k}} = F_{S_2\wr S_k}\big\uparrow^{S_{2k}} \oplus \left(F_{S_2\wr S_k}\big\uparrow^{S_{2k}}\right) \otimes \mathrm{sgn}_{S_{2k}}\,.$$

We may now apply Lemma 10 and Proposition 13 to deduce that the summands of the left-hand side have Specht filtrations.

It only remains to deal with the permutation modules from case (e). By our usual arguments, together with the remark following Theorem 11, it suffices to show that for $k = 5$, 6 or 9, $F\uparrow_{G_k}^{S_k}$ has a Specht filtration. This is immediate if $p > k$, as then every FS_k-module is a direct sum of Specht modules. The other cases turn out to be surprisingly easy.

If $p = 5$ and $k = 5$ or $k = 6$ then the Specht modules corresponding to the non-trivial ordinary characters in $1_{G_k}\uparrow^{S_k}$ are simple and projective. Hence

$$F\big\uparrow_{G_5}^{S_5} \cong S^{(5)} \oplus S^{(2,2,1)} \quad \text{and} \quad F\big\uparrow_{G_6}^{S_6} \cong S^{(6)} \oplus S^{(2,2,2)}.$$

When $p = 5$ and $k = 9$ we observe that since $\mathrm{P\Gamma L}(2,8)$ does not contain any elements of order 5, $F\big\uparrow_{G_9}^{S_9}$ is projective. It is well known (see [13]) that any projective module for a symmetric group has a Specht filtration. The only case left is when $p = 7$ and $k = 9$. One sees from (6) that each irreducible ordinary character appearing in $1_{G_9}\uparrow^{S_9}$ lies in a different 7-block of S_9, and that the only non-projective constituents are the trivial and sign representations. Hence

$$F\big\uparrow_{G_9}^{S_9} \cong S^{(9)} \oplus S^{(1^9)} \oplus S^{(4,4,1)} \oplus S^{(3,2,2,2)} \oplus S^{(5,1^4)}.$$

This completes the proof of Theorem 3.

4. Two counterexamples

4.1. Let F be an algebraically closed field of characteristic 3. We shall show that the module $M = F\!\uparrow_{\mathrm{P\Gamma L}(2,8)}^{S_9}$ is a counterexample to the conjecture that every permutation module has a Specht filtration. It would be interesting to collect further examples of permutation modules over fields of odd characteristic which do not have Specht filtrations—at the moment, it is far from clear how common they are.

The shortest demonstration that the author has been able to discover hinges on the simple module $D^{(5,2,2)}$. (See [12, Definition 11.2] for the definition of the D^μ.) One can show, either with the help of computer algebra, or more lengthily by *ad hoc* arguments (see below), that M contains a submodule isomorphic to $D^{(5,2,2)}$. It follows from the table of decomposition numbers of S_9 in characteristic 3 (see [12, p145]) that there are only four Specht modules which (a) have $D^{(5,2,2)}$ as a composition factor, and (b) do not also have other composition factors that are absent in M. They are

$$S^{(5,2,2)}, S^{(3,2^3)}, S^{(5,1^4)}, S^{(3,3,1^3)}.$$

By a standard result (see [12, Corollary 12.2]), $D^{(5,2,2)}$ only appears at the top of $S^{(5,2,2)}$. By (9), $D^{(5,2,2)}$ appears in the socle of $S^{(3,2^3)}$ if and only if $D^{(5,2,2)} \otimes \mathrm{sgn} = D^{(6,2,1)}$ appears in the top of $S^{(4,4,1)}$; this is also ruled out by [12, Corollary 12.2]. The same argument works for $S^{(3,3,1^3)}$. Finally, one can use the long exact sequence

$$S^{(9)} \to S^{(8,1)} \to S^{(7,1^2)} \to S^{(6,1^3)} \to S^{(5,1^4)} \to \cdots$$

given by the maps θ_r constructed by Hamernik in [6, p449] to show that $S^{(5,1^4)}$ contains $D^{(5,2,2)}$ in its top, but not in its socle. (Hamernik works only with symmetric groups of prime degree, but it is easy to generalise this part of his work to deal with hook-Specht modules for FS_n whenever the characteristic of F divides n: see [17, §1.3].) Thus none of the candidate Specht modules contains $D^{(5,2,2)}$ as a submodule. The result follows.

It remains to show that $D^{(5,2,2)}$ appears in the socle of M. For this we shall need the following lemma, which is of some independent interest.

Lemma 14. *Let F be an algebraically closed field, let G be a finite group and let M be an indecomposable FG-module such that*

(a) $\operatorname{soc} M \cong \operatorname{top} M \cong F$,

(b) *F appears exactly twice as a composition factor of M.*

Then $\mathrm{End}_{FG}(M)$ *is 2-dimensional.*

Proof. Let $\theta \in \mathrm{End}_{FG}(M)$. By a corollary of Fitting's Lemma (see [1, Lemma 1.4.5]), θ is either nilpotent or invertible. If θ is nilpotent then $M/\ker\theta$ is isomorphic to a proper submodule of M; by (b) this can only happen if $\ker\theta = \operatorname{rad} M$ and θ is, up to a scalar, the map ν defined by

$$M \twoheadrightarrow M/\operatorname{rad} M \cong F \hookrightarrow M.$$

If θ is invertible, then it has a non-zero eigenvalue $t \in F$. Since $\theta - t1_M$ is not invertible, it must be a scalar multiple of ν. Hence $\operatorname{End}_{FG}(M) = \langle 1_M, \nu\rangle_F$. □

Proposition 15. *Let F be an algebraically closed field of characteristic* 3. *The simple module $D^{(5,2,2)}$ lies in the socle of $F\!\uparrow_{\mathrm{P\Gamma L}(2,8)}^{S_9}$.*

Proof. By basic Clifford theory, it is equivalent to show that $D^{(5,2,2)}\!\downarrow_{A_9}$ lies in the socle of $N = F\!\uparrow_{\mathrm{P\Gamma L}(2,8)}^{A_9}$; for an introduction to the Clifford theory needed to relate representations of the alternating and symmetric groups, see [4, Chapter 5]. It follows from (6) in §2.3 that

$$1_{\mathrm{P\Gamma L}(2,8)}\!\uparrow^{A_9} = 1 + \chi^{(5,1^4)^+} + \chi^{(4,4,1)}\!\downarrow_{A_9}$$

where $\chi^{(5,1^4)^+}$ is one of the two irreducible constituents of $\chi^{(5,1^4)}\!\downarrow_{A_9}$. (The labelling of this pair of characters is essentially arbitrary, and we do not need to be any more precise here.) From this, one can use decomposition numbers of S_9 to show that the composition factors of N are

$$F, F, D^{(8,1)}\!\downarrow_{A_9}, D^{(6,3)}\!\downarrow_{A_9}, D^{(5,2,2)}\!\downarrow_{A_9}, D^{(5,2,2)}\!\downarrow_{A_9}.$$

Let Q be a Sylow 3-subgroup of $\mathrm{P\Gamma L}(2,8)$ and let P be a Sylow 3-subgroup of A_9 containing Q. Note that $|P : Q| = 3$. It follows from Mackey's lemma that

$$N\!\downarrow_P = \bigoplus_g F\!\uparrow_{\mathrm{P\Gamma L}^g \cap P}^{P} \tag{10}$$

where g runs over a set of representatives for the double cosets of $\mathrm{P\Gamma L}(2,8)$ and P in A_9. If R is any subgroup of P then, by Frobenius reciprocity

$$\dim \operatorname{Hom}_{FP}(F\!\uparrow_R^P, F) = \dim \operatorname{Hom}_{FR}(F, F) = 1.$$

Hence each summand of the right-hand side of (10) is indecomposable with a 1-dimensional socle. The dimension of $F\!\uparrow_{\mathrm{P\Gamma L}^g\cap P}^P$ is $|P : \mathrm{P\Gamma L}^g \cap P|$ which is divisible by $|P : Q| = 3$. It follows that if U is any indecomposable summand of N (considered as a FA_9-module) then U has dimension divisible by 3.

Since the ordinary character $1_{\mathrm{P\Gamma L}(2,8)}\!\uparrow^{A_9}$ is multiplicity-free with three irreducible constituents, the $\mathbf{Q}A_9$-permutation module $\mathbf{Q}\!\uparrow_{\mathrm{P\Gamma L}(2,8)}^{A_9}$ has a 3-dimensional endomorphism algebra. By [1, Theorem 3.11.3], the endomorphism algebra $\operatorname{End}_{FA_9}(N)$ is also 3-dimensional.

The 7-dimensional simple module $D^{(8,1)}\!\downarrow_{A_9}$ and the 41-dimensional simple module $D^{(6,3)}$ cannot appear as summands of N. Since they each appear but once as compositions factors of N, they must lie in its middle Loewy layer. If, in addition, $D^{(5,2,2)}\!\downarrow_{A_9}$ does not appear in the socle of N, then N

must be indecomposable, with top and socle both isomorphic to the trivial module F. But then Lemma 14 implies that $\mathrm{End}_{FA_9}(N)$ is 2-dimensional, a contradiction. Hence $\mathrm{soc}\, N$ contains $D^{(5,2,2)}{\downarrow_{A_9}}$. □

Remark: A small extension of this argument shows that $M = F{\uparrow}_{\mathrm{P\Gamma L}(2,8)}^{S_9}$ is indecomposable, with the Loewy layers shown below.

$$\begin{array}{c} F \oplus \mathrm{sgn} \oplus D^{(6,2,1)} \oplus D^{(5,2,2)} \\ D^{(8,1)} \oplus D^{(4,4,1)} \oplus D^{(6,3)} \oplus D^{(3,3,2,1)} \\ F \oplus \mathrm{sgn} \oplus D^{(6,2,1)} \oplus D^{(5,2,2)} \end{array}$$

4.2. We now consider $\mathbf{F}_2{\uparrow}_{S_3 \wr S_2}^{S_6}$. It is easy to show that this module has the Loewy layers

$$\begin{array}{c} \mathbf{F}_2 \\ D^{(5,1)} \oplus D^{(4,2)} \\ \mathbf{F}_2 \end{array} .$$

By (4) in §2.2, the ordinary character associated to $\mathbf{F}_2{\uparrow}_{S_3 \wr S_2}^{S_6}$ is $\chi^{(6)} + \chi^{(4,2)}$. It is known that although the trivial module is a composition factor of $S^{(4,2)}$, it does not appear in either the top or socle of $S^{(4,2)}$ (see [12, Example 24.5(iii)]). It follows that there is no Specht filtration of $\mathbf{F}_2{\uparrow}_{S_3 \wr S_2}^{S_6}$ with the factors $S^{(6)}$ and $S^{(4,2)}$. One can, however, exploit the outer automorphism of S_6, which sends the Specht module $S^{(5,1)}$ to $S^{(3,3)} = S^{(2,2,2)^\star}$ and leaves $\mathbf{F}_2{\uparrow}_{S_3 \wr S_2}^{S_6}$ fixed, to show that there is a short exact sequence

$$0 \rightarrow S^{(5,1)} \rightarrow \mathbf{F}_2{\uparrow}_{S_3 \wr S_2}^{S_6} \rightarrow S^{(2^3)} \rightarrow 0.$$

Thus $\mathbf{F}_2{\uparrow}_{S_3 \wr S_2}^{S_6}$ has a Specht filtration, but the Specht factors required are not those indicated by the associated ordinary character. It is left to the reader to formulate any of the many conjectures to which this module is a counterexample.

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References

[1] D. J. Benson, Representations and cohomology I, vol. 30 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1995.

[2] W. Bosma, J. Cannon, C. Playoust. The Magma algebra system. I. The user language. J. Symbolic Comput. 24 (1997) 235-265

[3] P. J. Cameron, Permutation groups, vol. 45 of London Math. Soc. Student Texts, Cambridge University Press, 1999.

[4] W. Fulton, J. Harris, Representation theory, a first course, vol. 129 of Graduate Texts in Mathematics, Springer, New York, 1991.

[5] C. Godsil, K. Meagher, Multiplicity-free representations of the symmetric group, To appear in Annals of Combinatorics, available from arXiv:math/0612567

[6] W. Hamernik, Specht modules and the radical of the group ring over the symmetric group γ_p, Comm. Algebra 4 (1976) 435-457.

[7] D. J. Hemmer, D. K. Nakano, Specht filtrations and Hecke algebras of Type A, J. London Math. Soc. 69 (2004) 623-638.

[8] D. J. Hemmer, Symmetric group modules with Specht and dual Specht filtrations, Comm. Algebra 35 (2007) 3292-3306.

[9] J. E. Humphreys, Introduction to Lie algebras and representation theory, vol. 9 of Graduate Texts in Mathematics, Springer, New York, 1978.

[10] N. F. Inglis, R. W. Richardson, J. Saxl, An explicit model for the complex representations of S_n, Arch. Math. (Basel) 54 (1990) 258-259.

[11] G. D. James, A. Kerber, The representation theory of the symmetric group, vol. 16 of Encyclopedia of Mathematics and its Applications, Addison-Wesley Publishing Co., Reading, Mass., 1981.

[12] G. D. James, The representation theory of the symmetric groups, vol. 682 of Lecture Notes in Mathematics, Springer, Berlin, 1978.

[13] G. D. James, Trivial source modules for symmetric groups, Arch. Math. (Basel) 41 (1983) 294-300.

[14] R. Paget, A family of modules with Specht and dual Specht filtrations. J. Algebra 312 (2007) 880-890.

[15] J. Saxl, On multiplicity-free permutation representations, in: Finite geometries and designs (Proc. Conf., Chelwood Gate, 1980), vol. 49 of *London Math. Soc. Lecture Note Ser.*, Cambridge University Press, 1981, pp. 337-353

[16] R. M. Thrall, On symmetrized Kronecker powers and the structure of the free Lie ring, Amer. J. Math. 64 (1942) 371-388.

[17] M. Wildon, Modular representations of symmetric groups, D. Phil Thesis, University of Oxford, 2004.

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