A NEW BIJECTIVE PROOF OF THE q-PFAFF–SAALSCHÜTZ IDENTITY WITH APPLICATIONS TO QUANTUM GROUPS

ÁLVARO GUTIÉRREZ, ÁLVARO L. MARTÍNEZ,
MICHAŁ SZWEJ, AND MARK WILDON

ABSTRACT. We present a bijective proof of a family of q-binomial identities — including Stanley's identity — each equivalent to the q-Pfaff–Saalschütz identity. Our proof is far shorter than those previously known and is the first in which q-binomial coefficients are interpreted as counting subspaces of $\mathbb{F}_q$-vector spaces. As a corollary, we obtain a new multiplication rule for quantum binomial coefficients and hence a new presentation of Lusztig's integral form $\mathcal{U}_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{sl}_2)$ of the Cartan subalgebra of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$.

1. INTRODUCTION

The study of binomial identities of the form $\binom{\cdot}{\cdot}\binom{\cdot}{\cdot} = \sum \binom{\cdot}{\cdot}\binom{\cdot}{\cdot}\binom{\cdot}{\cdot}$ dates back as early as 1303 to an identity stated by Shih-Chieh Chu [Tak73]. Multiple identities of this form were discovered around the 1970s by Nanjundiah [Nan58], Stanley [Sta70], Bizley [Biz70], Takács [Tak73] and Székely [Szé85]. In the setting of hypergeometric series [GKP89, §5.5], it becomes clear that each of them is an instance of the Pfaff–Saalschütz identity [Zen89] — in other words, they are all equivalent by a simultaneous linear change of variables. Each of these identities lifts to a q-analogue of the form $\left[{\cdot \atop \cdot}\right]_q \left[{\cdot \atop \cdot}\right]_q = \sum q^{\cdot} \left[{\cdot \atop \cdot}\right]_q \left[{\cdot \atop \cdot}\right]_q \left[{\cdot \atop \cdot}\right]_q$, with the same equivalences holding. We may therefore refer to any identity of this form as a q-Pfaff–Saalschütz identity. The combinatorial proofs to date of q-Pfaff–Saalschütz identities are due to Andrews and Bressoud [AB84], Goulden [Gou85], Zeilberger [Zei87], Yee [Yee08], and Schlosser and Yoo [SY17].

Let $\left[{n \atop k}\right]_q$ denote the q-binomial coefficient, defined for $n,\ k \in \mathbb{N}_0$ as a polynomial in q in §2. In §3 we present a new bijective proof of the q-Pfaff–Saalschütz identity in the following form.

Theorem 1.1 (q-Pfaff–Saalschütz identity)**.** *Let $m, s, t \in \mathbb{N}_0$. If $e \in \mathbb{Z}$ and $-t \le e \le s$ then*

$$\begin{bmatrix} m \\ t \end{bmatrix}_q \begin{bmatrix} m+e \\ s \end{bmatrix}_q = \sum_{j \geqslant 0} q^{(s-j)(t+e-j)} \begin{bmatrix} t+e \\ j \end{bmatrix}_q \begin{bmatrix} s-e \\ s-j \end{bmatrix}_q \begin{bmatrix} m+j \\ s+t \end{bmatrix}_q .$$

Our proof is shorter than the ones existing in the literature, and relies on a combinatorial interpretation of the q-binomial coefficients not previous used in this context: if q is a prime power, then $\left[{n \atop k}\right]_q$ is the number of k-dimensional subspaces of $\mathbb{F}_q^n$ [KC02, Thm. 7.1].

The second main result of this paper lifts Theorem 1.1 to an identity in the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. In [Lus88], Lusztig defines an integral form $\mathcal{U}_{\mathcal{A}}(\mathfrak{sl}_2)$ for $\mathcal{U}_q(\mathfrak{sl}_2)$ over $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. It can be thought of as a deformation of Kostant's $\mathbb{Z}$-form for $\mathcal{U}(\mathfrak{sl}_2)$, and it admits a triangular decomposition

$$\mathcal{U}_{\mathcal{A}}(\mathfrak{sl}_2) = \mathcal{U}^-_{\mathcal{A}}(\mathfrak{sl}_2) \otimes_{\mathcal{A}} \mathcal{U}^0_{\mathcal{A}}(\mathfrak{sl}_2) \otimes_{\mathcal{A}} \mathcal{U}^+_{\mathcal{A}}(\mathfrak{sl}_2).$$

To construct the subalgebra $\mathcal{U}^0_{\mathcal{A}}(\mathfrak{sl}_2)$, Lusztig uses the elements

$$\begin{bmatrix} K; c \\ t \end{bmatrix} = \frac{[K; c][K; c-1] \cdots [K; c-t+1]}{\{t\}_q!}, \tag{1.1}$$

for $c \in \mathbb{Z}$ and $t \in \mathbb{N}_0$, where $[K; a] = \frac{q^a K - q^{-a} K^{-1}}{q - q^{-1}}$ and $\{t\}_q!$ is the quantum factorial defined in §4. A lift of the identity in Theorem 1.1 to the quantum group gives the multiplication rule for these elements. In the following theorem, $\left\{ n \atop k \right\}_q$ denotes the quantum binomial coefficient, defined as a Laurent polynomial in q in §4.

Theorem 1.2. *Let $b, c \in \mathbb{Z}$ and $s, t \in \mathbb{N}_0$. If $t - c + b \geqslant 0$ and $s - b + c \geqslant 0$, then the elements defined in* (1.1) *satisfy the following multiplication rule:*

$$\begin{bmatrix} K; c \\ t \end{bmatrix} \begin{bmatrix} K; b \\ s \end{bmatrix} = \sum_{i \geq 0} \begin{Bmatrix} t - c + b \\ i - c \end{Bmatrix}_q \begin{Bmatrix} s - b + c \\ i - b \end{Bmatrix}_q \begin{bmatrix} K; i \\ t + s \end{bmatrix}.$$

Lusztig shows that the elements $K^\delta \left[K \atop t \right]$ for $t \geq 0$ and $\delta \in \{0, 1\}$ form an $\mathcal{A}$-basis for $\mathcal{U}^0_{\mathcal{A}}(\mathfrak{sl}_2)$. In Proposition 5.3, we show that the elements $\left[K; c \atop t \right]$ for $t \geq 0$ and $c \in \{0, 1\}$ also form an $\mathcal{A}$-basis for $\mathcal{U}_{\mathcal{A}}(\mathfrak{sl}_2)$. This gives our new description of the multiplication in $U^0_{\mathcal{A}}(\mathfrak{sl}_2)$ in Theorem 6.2. As a corollary, we obtain a presentation of $U^0_{\mathcal{A}}(\mathfrak{g})$ for an arbitrary Kac–Moody algebra $\mathfrak{g}$ of finite rank: see Corollary 6.4.

Outline. The structure of the rest of this paper is as follows. In §2 we give preliminary results about q-binomial coefficients, including bijective proofs of certain basic identities using the vector space interpretation. In §3 we present our new bijective proof of Theorem 1.1. In §4 we give the quantum version of this identity, later used to prove Theorem 1.2. The quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ and Lusztig's integral form are briefly reviewed in §5. Finally in §6 we prove Theorem 1.2 and, as a corollary, we obtain a new presentation of Lusztig's integral form.

2. q-binomials and subspaces of $\mathbb{F}_q^n$

In this section, we state foundational results on q-binomial coefficients. Certain proofs are included both for completeness and to familiarize the reader with the vector space interpretation of q-binomial coefficients. These tools will be instrumental in §3, where we present the new proof of Theorem 1.1. We refer the reader to [KC02] for further background.

We define the *q-integer* by

$$[n]_q = q^{n-1} + \cdots + q + 1 = \frac{q^n - 1}{q - 1}$$

for $n \in \mathbb{N}_0$. The definitions of q-factorial and q-binomial coefficient follow naturally: $[n]_q! = [n]_q[n-1]_q \dots [1]_q$ and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$$

for $n, k \in \mathbb{N}_0$. We set $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ whenever $n < k$.

Remark 2.1. Under this convention, the sum in Theorem 1.1 is finite, with non-zero summands for $\max\{0, e\} \leqslant j \leqslant \min\{t + e, s\}$.

The q-binomial coefficients have numerous combinatorial interpretations: as generating functions of subsets of $\mathbb{N}_0$ by the sum of their elements, partitions in a $(n-k) \times k$ rectangle, inversions in permutations, an so on [Sta12, Chapter 1]. In the following proposition and for the remainder of this section, q is a prime power.

Proposition 2.2. *The number of k-dimensional subspaces of $\mathbb{F}_q^n$ equals $\begin{bmatrix} n \\ k \end{bmatrix}_q$.*

Proof. See [KC02, Thm. 7.1]. □

Example 2.3. Let $n = 2$, $k = 1$. By definition, $\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = \frac{q^2-1}{q-1} = q + 1$. When $q = 3$, there are indeed four lines through the origin in the projective plane $\mathbb{F}_3^2$, as shown in Figure 1.

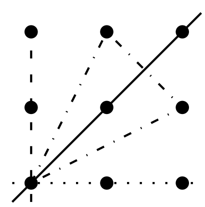


FIGURE 1. The four one-dimensional subspaces of $\mathbb{F}_3^2$, visualizing $\mathbb{F}_3^2$ as a subset of the plane.

This interpretation allows to lift bijective proofs of classical binomial identities, in which $\binom{n}{k}$ counts the number of k-subsets of $\{1, \dots, n\}$, into bijective proofs of their q-analogues. As an illustrative example, we give a bijective proof of the following basic identity, used later in the proof of Theorem 1.1. Throughout we use $\leqslant$ for containment of subspaces.

Lemma 2.4. *Let $k \leqslant \ell \leqslant n$ be non-negative integers. Then*

$$\begin{bmatrix} n \\ \ell \end{bmatrix}_q \begin{bmatrix} \ell \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n-k \\ \ell-k \end{bmatrix}_q.$$

Proof. If $k = 0$, the identity holds trivially. Assume now that $k \geqslant 1$. By Proposition 2.2, the left-hand side is the number of pairs of subspaces (U, W) of $V = \mathbb{F}_q^n$ satisfying $U \leqslant W \leqslant V$, $\dim U = k$ and $\dim W = \ell$.

Similarly, the right-hand side counts the number of pairs $(U, \overline{W})$, where U is a k-dimensional subspace of V and $\overline{W}$ is an $(\ell - k)$-dimensional subspace of the $(n-k)$-dimensional quotient space V/U.

The natural bijection between these two sets given by

$$(U, W) \mapsto (U, W/U)$$

proves that they are of equal size. □

While some q-binomial identities are very similar to their binomial specializations, in general a more careful treatment is required. One of the main reasons for this is the non-uniqueness of complements to a subspace of a vector space.

Proposition 2.5. *Let $k \leqslant n$ be non-negative integers. If U is a k-dimensional subspace of $\mathbb{F}_q^n$ then U has $q^{k(n-k)}$ distinct complements inside $\mathbb{F}_q^n$.*

Proof. We will construct a basis of a complement of U. As the first vector of the basis, choose any w_1 outside of U — this can be done in $q^n - q^k$ ways. Similarly, for w_2 choose any vector which does not belong to $U \oplus \langle w_1 \rangle_{\mathbb{F}_q}$ — this can be done in $q^n - q^{k+1}$ many ways. Repeating this process $n - k$ times, we construct

$$(q^n - q^k)(q^n - q^{k+1}) \ldots (q^n - q^{n-1})$$

many bases, each spanning a complement. However, each complement is obtained in

$$(q^{n-k} - 1)(q^{n-k} - q) \ldots (q^{n-k} - q^{n-k-1})$$

many ways. The number of distinct complements of U in $\mathbb{F}_q^n$ is therefore the quotient of the two quantities displayed above, namely $q^{k(n-k)}$. □

Example 2.6. As seen in Figure 1, when $q = 3$, $n = 2$ and $k = 1$ each line has 3 distinct complements in the plane $\mathbb{F}_3^2$.

The second basic identity needed in the proof of Theorem 1.1 is as follows. Again we give a bijective proof.

Lemma 2.7. *Let $k \leqslant n$ be non-negative integers. Then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q.$$

Proof. Let V be an n-dimensional $\mathbb{F}_q$-vector space. The map $U \mapsto \{\theta \in V^\star : \theta(U) = 0\}$ is a bijection between k-dimensional subspaces of V and the $(n-k)$-dimensional subspaces of the dual space $V^\star$. □

Another important difference when compared to binomial coefficients and subsets is the analogue of set union. Indeed, by taking a direct sum of two subspaces we *lose* information about the vectors which belong to neither of the direct summands. A more careful approach, frequently used in our proof of Theorem 1.1 is through *extension*.

Proposition 2.8. *Let k, ℓ, m, n be non-negative integers satisfying $\ell \leqslant k \leqslant n$ and $\ell \leqslant m \leqslant n$. Let V_k be a k-dimensional subspace of $V = \mathbb{F}_q^n$ and V_ℓ an ℓ-dimensional subspace of V_k. The number of distinct m-dimensional extensions V_m of V_ℓ inside of V, such that $V_m \cap V_k = V_\ell$, equals*

$$q^{(m-\ell)(k-\ell)} \begin{bmatrix} n-k \\ m-\ell \end{bmatrix}_q .$$

Proof. See [KC02, p. 23] for a bijective proof. □

Example 2.9. If $k = \ell$, the extensions V_m are in a one-to-one correspondence with the $(m-\ell)$-dimensional subspaces of the quotient space V/V_k. The formula in Proposition 2.8 correctly simplifies to $\begin{bmatrix} n-k \\ m-k \end{bmatrix}_q$. If instead $\ell = 0$ and $m = n - k$, then extensions V_m are precisely the complements of V_k in V. The formula in Proposition 2.8 simplifies to $q^{(n-k)k}$, in agreement with Proposition 2.5.

The third basic identity needed in the proof of Theorem 1.1 is a q-lift of Vandermonde's convolution. Again it has a bijective proof.

Proposition 2.10 (q-Vandermonde's convolution)**.** *Let $m \leqslant n$ be non-negative integers. Then*

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \sum_{\ell=0}^{m} q^{\ell(n-k-m+\ell)} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \begin{bmatrix} n-k \\ m-\ell \end{bmatrix}_q$$

for any non-negative integer $k \leqslant n$.

Proof. In [KC02, p. 23] the authors give a simple bijective proof that

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \sum_{\ell=0}^{m} q^{(m-\ell)(k-\ell)} \begin{bmatrix} k \\ \ell \end{bmatrix}_q \begin{bmatrix} n-k \\ m-\ell \end{bmatrix}_q$$

Now change variables by $\ell \mapsto m - \ell$ and $k \mapsto n - k$ and apply Lemma 2.7 (which we proved bijectively) to get the result. □

Remark 2.11. Since the q-integers are polynomials in q with coefficients in $\mathbb{C}$, to prove a q-binomial identity it suffices to prove it for infinitely many values of q, such as the prime powers. In particular, the identities obtained in this section hold for all $q \in \mathbb{C} \setminus \{0\}$.

3. A new bijective proof of Theorem 1.1

In this section we construct the bijective map that establishes Theorem 1.1 — the main difficulty is to construct a suitable domain and codomain. We observe immediately that the left-hand side of Theorem 1.1 counts pairs of subspaces of dimensions s and t in $\mathbb{F}_q^m$ and $\mathbb{F}_q^{m+e}$, respectively. The right-hand side requires a more detailed description. A big step toward this goal is the following lemma. In it we use $X^{\perp Y}$ to denote an arbitrary vector space complement of X inside Y, and U_X a *fixed* subspace of X.

Lemma 3.1. *Let m, s, t, e, j, k be non-negative integers with $s-e \geqslant 0$ and $t+e-k \geqslant 0$. Let W be an m-dimensional subspace of $V = \mathbb{F}_q^{m+e}$. The number of septuples*

$$(A, B, C, U_C, D, A^{\perp C}, A^{\perp W}),$$

satisfying the following conditions:

- $\dim A = t$, $\dim B = s$, $\dim C = s+t-k$, $\dim D = j-e$, *and* $\dim U_C = s-e$,
- $A \leqslant C \leqslant W$, $D \leqslant U_C \leqslant C \leqslant W$, $B \leqslant V$, *and* $A^{\perp C} \leqslant A^{\perp W} \leqslant W$,
- $A \cap U_C = D$, $B \cap A^{\perp W} = A^{\perp C}$, *and* $A^{\perp W} \cap C = A^{\perp C}$

equals

$$q^{t(m-t)} q^{(s-j)(t+e-j)} q^{(m-s-t+k)k} \begin{bmatrix} t+e \\ k \end{bmatrix}_q \begin{bmatrix} t+e-k \\ j-k \end{bmatrix}_q \begin{bmatrix} s-e \\ s-j \end{bmatrix}_q \begin{bmatrix} m \\ s+t-k \end{bmatrix}_q.$$

The conditions, the choice of septuple, as well as the formula, arise naturally in the process of construction in the proof below.

Proof of Lemma 3.1. We illustrate the proof in Figure 2, building on the set-up in Figure 3a. Choose an $(s+t-k)$-dimensional subspace C of W (Figure 3b) in $\left[\begin{smallmatrix} m \\ s+t-k \end{smallmatrix}\right]_q$ many ways and *fix* an a subspace U_C of C of dimension $s-e$ (Figure 3c). Next, choose a $(j-e)$-dimensional subspace D of U_C (Figure 3d) in $\left[\begin{smallmatrix} s-e \\ j-e \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} s-e \\ s-j \end{smallmatrix}\right]_q$ many ways. By Proposition 2.8, the subspace D of dimension $j-e$ can be extended to a t-dimensional subspace A of C (Figure 3e), such that $A \cap U_C = D$, in $q^{(s-j)(t+e-j)} \left[\begin{smallmatrix} t+e-k \\ j-k \end{smallmatrix}\right]_q$ many ways. Thus, we obtain

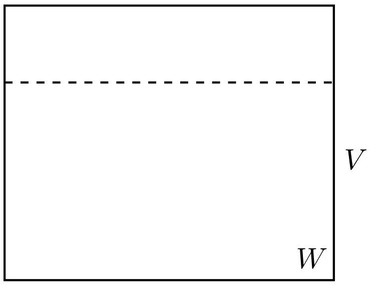
$$q^{(s-j)(t+e-j)} \begin{bmatrix} t+e-k \\ j-k \end{bmatrix}_q \begin{bmatrix} s-e \\ s-j \end{bmatrix}_q \begin{bmatrix} m \\ s+t-k \end{bmatrix}_q$$

many triples (A, C, D) of subspaces of W, satisfying:

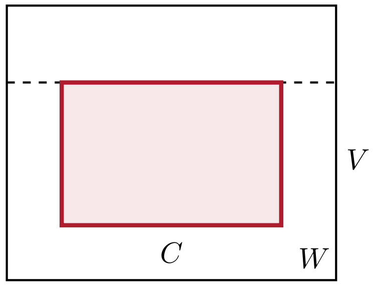
- $\dim A = t$, $\dim C = s+t-k$, and $\dim D = j-e$,
- $A \leqslant C \leqslant W$, $D \leqslant U_C \leqslant C \leqslant W$,
- $A \cap U_C = D$.

Let $A^{\perp C}$ be a complement of A in C (Figure 3f) — using Proposition 2.5 we can choose it in $q^{t(s-k)}$ many ways. By Proposition 2.8, the subspace $A^{\perp C}$ of

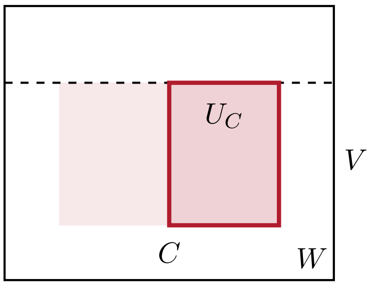
FIGURE 2. The construction of septuples in diagrams



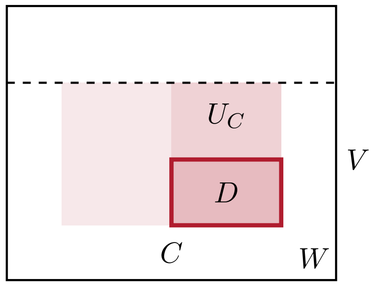
(a) Fix $W \cong \mathbb{F}_q^m$ inside $V \cong \mathbb{F}_q^{m+e}$.



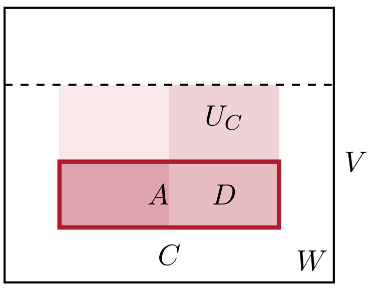
(b) Choose $C \cong \mathbb{F}_q^{s+t-k}$ inside W in $\begin{bmatrix} m \\ s+t-k \end{bmatrix}_q$ many ways.



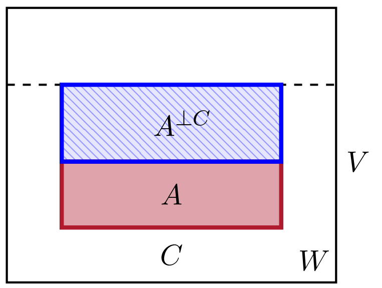
(c) Fix $U_C \cong \mathbb{F}_q^{s-e}$ inside C.



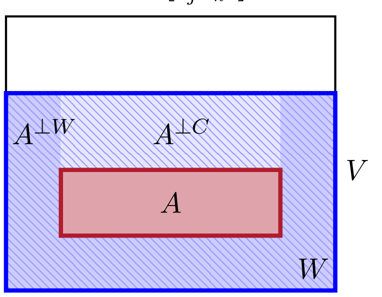
(d) Choose $D \cong \mathbb{F}_q^{j-e}$ inside U_C in $\begin{bmatrix} s-e \\ s-j \end{bmatrix}_q$ many ways.



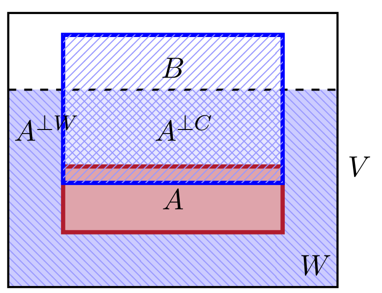
(e) Extend D to $A \cong \mathbb{F}_q^t$ inside C and outside of U_C in $q^{(s-j)(t+e-j)} \begin{bmatrix} t+e-k \\ j-k \end{bmatrix}$ many ways.



(f) Choose a complement $A^{\perp C} \cong \mathbb{F}_q^{s-k}$ of A inside C in $q^{t(s-k)}$ many ways.



(g) Extend $A^{\perp C}$ to $A^{\perp W} \cong \mathbb{F}_q^{m-t}$ inside W and outside of C in $q^{t(m-s-t+k)}$ many ways.



(h) Extend $A^{\perp C}$ to $B \cong \mathbb{F}_q^s$ inside V and outside of $A^{\perp W}$ in $q^{k(m-s-t+k)} \begin{bmatrix} t+e \\ k \end{bmatrix}_q$ many ways.

dimension $s-k$ can be extended to an $(m-t)$-dimensional vector space $A^{\perp W}$ in W (Figure 3g), such that $A^{\perp W} \cap C = A^{\perp C}$, in $q^{t(m-s-t+k)}$ many ways. This extension, as the notation already suggests, is a complement of A in W. Indeed, $A^{\perp W} \cap A = A^{\perp W} \cap (C \cap A) = (A^{\perp W} \cap C) \cap A = A^{\perp C} \cap A = 0$ and $\dim A^{\perp W} + \dim A = (m-t) + t = m = \dim W$.

From Proposition 2.8 applied to $A^{\perp C}$ as a subspace of $A^{\perp W}$, the $(s-k)$-dimensional vector space $A^{\perp C}$ can be extended to an s-dimensional vector space B in V (Figure 3h), such that $B \cap A^{\perp W} = A^{\perp C}$, in $q^{k(m-s-t+k)}\left[{t+e \atop k}\right]_q$ many ways. By construction, the vector spaces $A^{\perp C}$, $A^{\perp W}$ and B satisfy the conditions:

- $\dim B = s$,
- $B \leqslant V$, $A^{\perp C} \leqslant A^{\perp W} \leqslant W$,
- $B \cap A^{\perp W} = A^{\perp C}$, $A^{\perp W} \cap C = A^{\perp C}$.

We have therefore chosen a septuple satisfying the specified conditions, and the total number of choices is the product of the number of choices we made at each step; this gives the desired expression. □

To prove Theorem 1.1 when $e < 0$, we need the following standard result.

Lemma 3.2. *Let $f \in \mathbb{Q}(q)[X]$ have degree d as a polynomial in X. If $f(q^h) = 0$ for at least $d+1$ values of $h \in \mathbb{N}_0$, then $f = 0$. Moreover, if $g \in \mathbb{Q}(q)[X, X^{-1}]$ vanishes at (q^h, q^{-h}) for infinitely many values of $h \in \mathbb{N}_0$, then $g = 0$.*

Proof. The first part is a well-known fact about polynomials with coefficients in an integral domain. For the second part, take A sufficiently large that $X^A g(X, X^{-1})$ is a polynomial in X. The conclusion follows from the first part. □

Proof of Theorem 1.1 (q-Pfaff–Saalschütz identity). Assume $0 \leqslant e \leqslant s$. To simplify the right-hand side of Theorem 1.1, we apply the q-Vandermonde's convolution (Proposition 2.10) to the term $\left[{m+j \atop s+t}\right]_q$, followed by Lemma 2.4 to the product $\left[{t+e \atop j}\right]_q \left[{j \atop k}\right]_q$:

$$\begin{aligned}
&\sum_{j\geqslant 0} q^{(s-j)(t+e-j)} \begin{bmatrix} t+e \\ j \end{bmatrix}_q \begin{bmatrix} s-e \\ s-j \end{bmatrix}_q \begin{bmatrix} m+j \\ s+t \end{bmatrix}_q \\
&= \sum_{j\geqslant 0} q^{(s-j)(t+e-j)} \begin{bmatrix} t+e \\ j \end{bmatrix}_q \begin{bmatrix} s-e \\ s-j \end{bmatrix}_q \sum_{k\geqslant 0} q^{(m-s-t+k)k} \begin{bmatrix} m \\ s+t-k \end{bmatrix}_q \begin{bmatrix} j \\ k \end{bmatrix}_q \\
&= \sum_{j\geqslant 0} \sum_{k\geqslant 0} q^{(s-j)(t+e-j)} q^{(m-s-t+k)k} \begin{bmatrix} t+e \\ k \end{bmatrix}_q \begin{bmatrix} t+e-k \\ j-k \end{bmatrix}_q \begin{bmatrix} s-e \\ s-j \end{bmatrix}_q \begin{bmatrix} m \\ s+t-k \end{bmatrix}_q .
\end{aligned}$$

These two steps encode bijections from Proposition 2.10 and Lemma 2.4. Hence, upon further multiplying both sides by $q^{t(m-t)}$, the identity from Theorem 1.1 is bijectively equivalent to

$$q^{t(m-t)}\begin{bmatrix}m\\t\end{bmatrix}_q\begin{bmatrix}m+e\\s\end{bmatrix}_q = q^{t(m-t)}\sum_{j\geqslant 0}\sum_{k\geqslant 0} q^{(s-j)(t+e-j)}q^{(m-s-t+k)k}\begin{bmatrix}t+e\\k\end{bmatrix}_q \begin{bmatrix}t+e-k\\j-k\end{bmatrix}_q\begin{bmatrix}s-e\\s-j\end{bmatrix}_q\begin{bmatrix}m\\s+t-k\end{bmatrix}_q. \tag{3.1}$$

Now let q be a prime power. Let W be a fixed m-dimensional subspace of $V=\mathbb{F}_q^{m+e}$. Consider the set $\mathcal{T}$ of triples $(A,B,A^{\perp W})$ of subspaces of V satisfying

- $\dim A = t$ and $\dim B = s$,
- $A \leqslant W$ and $B \leqslant V$.

By Proposition 2.2 and Proposition 2.5, $|\mathcal{T}|$ is the left-hand side of equality (3.1). Let $\mathcal{S}_{j,k}$ denote the set of septuples described in Lemma 3.1. Define $\mathcal{S}=\bigcup_{j,k\in\mathbb{N}}\mathcal{S}_{j,k}$. By Lemma 3.1, $|\mathcal{S}|$ is the right-hand side of (3.1).

It now sufficient to exhibit a bijection between the sets $\mathcal{T}$ and $\mathcal{S}$. The bijection follows directly from the step-by-step construction detailed above (see also Figure 3h).

One direction of the bijection is the natural projection $\mathcal{S}\to\mathcal{T}$ defined by

$$(A,B,C,U_C,D,A^{\perp C},A^{\perp W})\mapsto (A,B,A^{\perp W}).$$

For the inverse, recall the conditions in the third bullet point of Lemma 3.1. We will show that the triple uniquely determines the corresponding septuple. Indeed, since $A^{\perp C}=B\cap A^{\perp W}$, we have $C=A\oplus A^{\perp C}=A\oplus(B\cap A^{\perp W})$. Hence $U_C=U_{A\oplus(B\cap A^{\perp W})}$ (recall that U_C was defined as a fixed subspace of C) and $D=A\cap U_C = A\cap U_{A\oplus(B\cap A^{\perp W})}$. Therefore, the map $\mathcal{T}\to\mathcal{S}$ defined by

$$(A,B,A^{\perp W})\mapsto \big(A,B,A\oplus(B\cap A^{\perp W}),U_{A\oplus(B\cap A^{\perp W})},A\cap U_{A\oplus(B\cap A^{\perp W})},B\cap A^{\perp W},A^{\perp W}\big)$$

is the desired inverse. This proves Theorem 1.1 for $0\le e\le s$ when q is a prime power, and the result for indeterminate q follows from Remark 2.11.

To prove the theorem for $-t\le e\le 0$, we define a function f by

$$f(X)=\begin{bmatrix}m\\t\end{bmatrix}_q\frac{(q^{m-s+1}X;q)_s}{(q;q)_s} - \sum_{j\geqslant 0} q^{(s-j)(t-j)}\frac{(q^{t-j+1}X;q)_j}{(q;q)_j}\cdot\frac{X^{s-j}(q^{j+1}X^{-1};q)_{s-j}}{(q;q)_{s-j}}\cdot\begin{bmatrix}m+j\\s+t\end{bmatrix}_q$$

where $(a;q)_\alpha$ denotes the shifted factorial product

$$(a;q)_\alpha=(1-a)(1-qa)\dots(1-q^{\alpha-1}a).$$

Note that $\begin{bmatrix} n \\ k \end{bmatrix}_q = (q^{n-k+1}; q)_k/(q;q)_k$ and hence

$$f(q^e) = \begin{bmatrix} m \\ t \end{bmatrix} \begin{bmatrix} m+e \\ s \end{bmatrix} - \sum_{j \geq 0} q^{(s-j)(t-j)} \begin{bmatrix} t+e \\ j \end{bmatrix} q^{e(s-j)} \begin{bmatrix} s-e \\ j \end{bmatrix} \begin{bmatrix} m+j \\ s+j \end{bmatrix}$$

which is the difference of the two sides in Theorem 1.1. By the combinatorial argument above we have $f(q^e) = 0$ for all $e \in \{0, 1, \ldots, s\}$. Since $f(X) \in \mathbb{Q}(q)[X]$ and $\deg f \leqslant s$, Lemma 3.2 implies that $f = 0$. In particular, $f(q^e) = 0$ for all $e \in \mathbb{Z}$ such that $-t \leq e \leq s$.

□

Corollary 3.3 (Pfaff–Saalschütz identity)**.** *Let* $m, s, t \in \mathbb{N}_0$. *If* $e \in \mathbb{Z}$ *and* $-t \leq e \leq s$ *then*

$$\binom{m}{t}\binom{m+e}{s} = \sum_{j \geqslant 0} \binom{t+e}{j}\binom{s-e}{s-j}\binom{m+j}{s+t}.$$

Proof. Take $q \to 1$ in Theorem 1.1. □

Remark 3.4. In the binomial case, one may consider A, B, C, D, U_C, V, W as sets and subsets instead of vector spaces and subspaces. Replacing $A^{\perp C}$ and $A^{\perp W}$ with the set-theoretic differences $C \setminus A$ and $W \setminus A$, respectively, the steps in the proof of Theorem 1.1 simplify significantly, giving a new bijective proof of the binomial identity in Corollary 3.3.

To illustrate the generality of Theorem 1.1, we show it is equivalent to Stanley's identity; this makes clear a symmetry that is hidden in our statement of Theorem 1.1.

Example 3.5 (Stanley's q-binomial identity)**.** Let $x, y, A, B \in \mathbb{N}$. Then

$$\begin{bmatrix} x+A \\ B \end{bmatrix}_q \begin{bmatrix} y+B \\ A \end{bmatrix}_q = \sum_{K \geqslant 0} q^{(A-K)(B-K)} \begin{bmatrix} x+y+K \\ K \end{bmatrix}_q \begin{bmatrix} y \\ A-K \end{bmatrix}_q \begin{bmatrix} x \\ B-K \end{bmatrix}_q$$

is equivalent to Theorem 1.1 via the simultaneous substitutions

$$(m, e, s, t, j) \mapsto (B+y, A+x-B-y, A+x-B, B+y-A, K-B+x),$$
$$(A, B, x, y, K) \mapsto (m-t, m+e-s, t+e, s-e, m-s-t+j).$$

As we mentioned in the introduction, multiple q-identities of this form have been rediscovered over the years. In the language of hypergeometric series, it becomes clear that they are all equivalent to the q-Pfaff–Saalschütz formula.

Remark 3.6. The q-binomial identity in Theorem 1.1 corresponds, in the notation of hypergeometric series, to the q-Pfaff–Saalschütz formula

$${}_3\phi_2\left(\begin{matrix} q^{s-m-e}, q^{-t-e}, q^{-t} \\ q^{-m-t-e}, q^{s+1-t-e} \end{matrix} ; q, q\right) = \frac{(q^{-m};q)_t (q^{-s-t};q)_t}{(q^{-m-t-e};q)_t (q^{-s+e};q)_t},$$

which in full generality [Zen89, (5)] states that for all $a, b, c \in \mathbb{C}$ and $n \in \mathbb{N}_0$:

$$ {}_3\phi_2\left(\begin{matrix} a, b, q^{-n} \\ c, q^{1-n}ab/c \end{matrix}; q, q\right) = \frac{(c/a;q)_n (c/b;q)_n}{(c;q)_n (c/ab;q)_n}. $$

4. The quantum Pfaff–Saalschütz identity

Let $q \in \mathbb{C} \setminus \{0, \pm 1\}$. To lift the identity in Theorem 1.1 to Theorem 1.2, we first express its quantum analogue. We define the *quantum integer* by

$$ \{n\}_q = q^{n-1} + q^{n-3} + \cdots + q^{1-n} = \frac{q^n - q^{-n}}{q - q^{-1}} $$

for $n \in \mathbb{N}_0$. The definitions of *quantum factorial* and *quantum binomial coefficient* again follow naturally: $\{n\}_q! = \{n\}_q \{n-1\}_q \dots \{1\}_q$ and

$$ \left\{\begin{matrix} n \\ k \end{matrix}\right\}_q = \frac{\{n\}_q!}{\{k\}_q! \{n-k\}_q!}, $$

for $n, k \in \mathbb{N}_0$. We set $\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}_q = 0$ whenever $n < k$. As we have seen, the q-binomial coefficients admit multiple combinatorial interpretations. On the other hand, quantum binomial coefficients are often used in algebra: for example $\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}$ is the character of the representation $\bigwedge^k \operatorname{Sym}^{n-1} \mathbb{C}^2$ of $\mathrm{SL}_2(\mathbb{C})$. The connection between the two, established in the following two results is a fundamental bridge in algebraic combinatorics, connecting combinatorial enumeration, plethysms of symmetric functions, and representation theory.

Lemma 4.1. *Let $n \in \mathbb{N}_0$. Then:*

$$ [n]_{q^2} = q^{n-1} \{n\}_q. $$

Proof. This follows directly from the definitions:

$$ [n]_{q^2} = \frac{q^{2n} - 1}{q^2 - 1} = \frac{q^n}{q} \cdot \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} \{n\}_q. \qquad \square $$

Corollary 4.2. *Let $k \leqslant n$ be non-negative integers. Then*

$$ \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} = q^{k(n-k)} \left\{\begin{matrix} n \\ k \end{matrix}\right\}_q. $$

Proof. This is immediate from Lemma 4.1. $\square$

We can now state and prove the quantum version of Theorem 1.1.

Theorem 4.3 (Quantum Pfaff–Saalschütz identity)**.** *Let $m, s, t \in \mathbb{N}_0$. If $e \in \mathbb{Z}$ and $-t \leq e \leq s$ then*

$$ \left\{\begin{matrix} m \\ t \end{matrix}\right\}_q \left\{\begin{matrix} m+e \\ s \end{matrix}\right\}_q = \sum_{j \geqslant 0} \left\{\begin{matrix} t+e \\ j \end{matrix}\right\}_q \left\{\begin{matrix} s-e \\ s-j \end{matrix}\right\}_q \left\{\begin{matrix} m+j \\ s+t \end{matrix}\right\}_q. $$

Proof of Theorem 4.3. By Corollary 4.2, replacing q with q^2 in Theorem 1.1 gives the required identity up to a power of q. The routine calculation

$$q^{t(m-t)}q^{s(m+e-s)} = q^{2(s-j)(t+e-j)}q^{j(t+e-j)}q^{(s-j)(j-e)}q^{(s+t)(m+j-s-t)},$$

shows that the powers of q cancel on both sides. □

The form of Theorem 4.3 we need for the proof of Theorem 1.2 is as follows.

Corollary 4.4. *Let $h, s, t \in \mathbb{N}_0$. If $b, c \in \mathbb{Z}$, $t - c + b \geqslant 0$, and $s - b + c \geqslant 0$ then*

$$\left\{ \begin{matrix} h+c \\ t \end{matrix} \right\}_q \left\{ \begin{matrix} h+b \\ s \end{matrix} \right\}_q = \sum_{i \geqslant 0} \left\{ \begin{matrix} t-c+b \\ i-c \end{matrix} \right\}_q \left\{ \begin{matrix} s-b+c \\ i-b \end{matrix} \right\}_q \left\{ \begin{matrix} h+i \\ t+s \end{matrix} \right\}_q.$$

Proof. Make the substitution $(m, e, j) \mapsto (h+c, b-c, i-c)$ in Theorem 4.3. The top entries in Theorem 4.3 are non-negative by assumption, and they remain non-negative after the substitution. □

5. Lusztig's integral form of $\mathcal{U}_q(\mathfrak{sl}_2)$

The quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ is the unital associative algebra over $\mathbb{Q}(q)$ with generators E, F, K, K^{-1} subject to the relation

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

and the expected $KK^{-1} = K^{-1}K = 1$. By Theorem 3.1.5 in [HK02] it has a triangular decomposition as

$$\mathcal{U}_q(\mathfrak{sl}_2) = \mathcal{U}_q^-(\mathfrak{sl}_2) \otimes_{\mathbb{Q}(q)} \mathcal{U}_q^0(\mathfrak{sl}_2) \otimes_{\mathbb{Q}(q)} \mathcal{U}_q^+(\mathfrak{sl}_2)$$

where $U_q^-(\mathfrak{sl}_2)$ (resp. $\mathcal{U}_q^0(\mathfrak{sl}_2)$, resp. $\mathcal{U}_q^+(\mathfrak{sl}_2)$) is the $\mathbb{Q}(q)$-subalgebra generated by F (resp. $K^{\pm 1}$, resp. E). Let

$$[K; a] = \frac{q^aK - q^{-a}K^{-1}}{q - q^{-1}}. \tag{5.1}$$

In [Lus88], Lusztig defines the *divided powers* $E^{(n)} = \frac{E^n}{\{n\}_q!}$ and $F^{(n)} = \frac{F^n}{\{n\}_q!}$, as well as the *Lusztig elements* $\left[\begin{smallmatrix} K;c \\ t \end{smallmatrix} \right]$ for $t \in \mathbb{N}_0$ and $c \in \mathbb{Z}$ already seen in (1.1). By definition $\left[\begin{smallmatrix} K;c \\ 0 \end{smallmatrix} \right] = 1$ and to simplify notation we set $\left[\begin{smallmatrix} K \\ t \end{smallmatrix} \right] = \left[\begin{smallmatrix} K;0 \\ t \end{smallmatrix} \right]$. Lusztig's element $\left[\begin{smallmatrix} K;c \\ t \end{smallmatrix} \right]$ should be thought of as a quantization of the element

$$\binom{H+c}{t} = \frac{(H+c)(H+c-1)\cdots(H+c-t+1)}{t!},$$

belonging to Kostant's $\mathbb{Z}$-form of the enveloping algebra of $\mathfrak{sl}_2$. Recall in the following definition that $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$.

Definition 5.1. *Lusztig's integral form* $\mathcal{U}_{\mathcal{A}}(\mathfrak{sl}_2)$ for $\mathcal{U}_q(\mathfrak{sl}_2)$ is the $\mathcal{A}$-algebra generated by the divided powers $E^{(n)} = \frac{E^n}{\{n\}_q!}$ and $\frac{F^n}{\{n\}_q!}$ for $n \in \mathbb{N}$ and the elements K, K^{-1} and $\left[\begin{smallmatrix} K \\ t \end{smallmatrix} \right]$ for $t \in \mathbb{N}$.

The divided powers satisfy the following relations for $n, m \in \mathbb{N}$:

$$E^{(n)}E^{(m)} = \left\{ {n+m \atop n} \right\}_q E^{(n+m)}, \tag{5.2}$$

$$F^{(n)}F^{(m)} = \left\{ {n+m \atop n} \right\}_q F^{(n+m)}. \tag{5.3}$$

We also have the following relations [Lus88, §4.3]:

$$\begin{bmatrix} K; c \\ t \end{bmatrix} E^{(n)} = E^{(n)} \begin{bmatrix} K; c+2n \\ t \end{bmatrix}, \tag{5.4}$$

$$\begin{bmatrix} K; c \\ t \end{bmatrix} F^{(n)} = F^{(n)} \begin{bmatrix} K; c-2n \\ t \end{bmatrix}, \tag{5.5}$$

$$E^{(n)}F^{(m)} = \sum_{t \geqslant 0} F^{(m-t)} \begin{bmatrix} K; 2t-m-n \\ t \end{bmatrix} E^{(n-t)}. \tag{5.6}$$

Proposition 5.2 (Lusztig). *The $\mathcal{A}$-algebras $\mathcal{U}_{\mathcal{A}}(\mathfrak{sl}_2)$ and $U^0_{\mathcal{A}}(\mathfrak{sl}_2)$ are free as $\mathcal{A}$-modules.*

(a) *The $\mathcal{A}$-algebra $\mathcal{U}_{\mathcal{A}}(\mathfrak{sl}_2)$ has an $\mathcal{A}$-basis given by the elements*

$$F^{(a)}K^{\delta}\begin{bmatrix} K \\ t \end{bmatrix}E^{(b)}$$

for $a, b, t \in \mathbb{N}_0$ and $\delta \in \{0, \min(1,t)\}$.

(b) *The elements $K^{\delta}\left[{K \atop t}\right]$ for $t \in \mathbb{N}_0$ and $\delta \in \{0, \min(1,t)\}$ form a basis for the Cartan subalgebra $U^0_{\mathcal{A}}(\mathfrak{sl}_2)$.*

Proof. The first part follows from Theorem 4.5 and Proposition 2.17 in [Lus90]. The second part follows from Theorem 4.5 and Proposition 2.14 in [Lus90]. □

We will now introduce a new $\mathcal{A}$-basis of Lusztig's integral form and its Cartan subalgebra. In the following section, we will combine these results with the multiplication rule from Theorem 1.2 to deduce a new presentation of these algebras.

Proposition 5.3. *Lusztig's integral form and its Cartan subalgebra admit the following bases:*

(1) *The $\mathcal{A}$-algebra $U^0_{\mathcal{A}}(\mathfrak{sl}_2)$ has an $\mathcal{A}$-basis*

$$\mathcal{B} = \left\{ \begin{bmatrix} K \\ t \end{bmatrix} : t \geq 0 \right\} \cup \left\{ \begin{bmatrix} K; 1 \\ t \end{bmatrix} : t \geq 1 \right\};$$

(2) *The elements $F^{(a)}\left[{K; c \atop t}\right]E^{(b)}$ for $a, b, t \in \mathbb{N}_0$ and $c \in \{0, \min(1,t)\}$, form an $\mathcal{A}$-basis of $U_{\mathcal{A}}(\mathfrak{sl}_2)$.*

Proof. First we compute using (5.1) that

$$q^t[K;1] - q^{-1}[K;-t] = q^t \frac{qK - q^{-1}K^{-1}}{q - q^{-1}} - q^{-1}\frac{q^{-t}K - q^t K^{-1}}{q - q^{-1}}$$

$$\begin{aligned} &= \frac{q^{t+1}K - q^{-(t+1)}K}{q-q^{-1}} \\ &= \{t+1\}_q K. \end{aligned}$$

Multiplying both sides by $[K;0]\cdots[K;-t+1]$ and dividing by $\{t+1\}_q!$, the right-hand side becomes $K\left[\begin{smallmatrix}K;0\\t\end{smallmatrix}\right]$, while the left-hand side becomes $q^t\left[\begin{smallmatrix}K;1\\t+1\end{smallmatrix}\right] - q^{-1}\left[\begin{smallmatrix}K;0\\t+1\end{smallmatrix}\right]$. Thus

$$K\begin{bmatrix}K\\t\end{bmatrix} = q^t\begin{bmatrix}K;1\\t+1\end{bmatrix} - q^{-1}\begin{bmatrix}K\\t+1\end{bmatrix}.$$

Since integer powers of q are units in $\mathcal{A}$, this gives an invertible change of basis from the basis claimed in (1) to the $\mathcal{A}$-basis of $\mathcal{U}^0_{\mathcal{A}}(\mathfrak{sl}_2)$ in Proposition 5.2(b). This proves (1) and (2) is a direct consequence of (1) and Proposition 5.2(a). $\square$

Remark 5.4. It is a notable feature of our basis $\mathcal{B}$ that it contains neither K nor K^{-1}. Instead, as a special case of Proposition 5.3, we may write

$$\begin{aligned} K &= \begin{bmatrix}K;1\\1\end{bmatrix} - q^{-1}\begin{bmatrix}K\\1\end{bmatrix}, \\ K^{-1} &= \begin{bmatrix}K;1\\1\end{bmatrix} - q\begin{bmatrix}K\\1\end{bmatrix}. \end{aligned}$$

6. The multiplication rule of Lusztig's elements $\left[\begin{smallmatrix}K;c\\t\end{smallmatrix}\right]$ and a new presentation of Lusztig's integral form $\mathcal{U}_{\mathcal{A}}(\mathfrak{sl}_2)$

In this section we present further new results about Lusztig's integral form and its Cartan subalgebra, including the multiplication rule from Theorem 1.2, and a new presentation of these algebras.

Proof of Theorem 1.2. We first unpack the definition of Lusztig's elements in (1.1):

$$\begin{bmatrix}K;c\\t\end{bmatrix} = \frac{[K;c][K;c-1]\cdots[K;c-t+1]}{\{t\}_q!} = \frac{\prod_{s=0}^{t-1}(Kq^{c-s} - K^{-1}q^{-(c-s)})}{\prod_{s=1}^{t}(q^s - q^{-s})}.$$

Observe that upon specializing K, K^{-1} to q^h, q^{-h}, respectively, Lusztig element specializes to a quantum binomial coefficient:

$$\begin{bmatrix}K;c\\t\end{bmatrix}\Bigg|_{\substack{K=q^h\\K^{-1}=q^{-h}}} = \frac{\prod_{s=0}^{t-1}(q^hq^{c-s} - q^{-h}q^{-(c-s)})}{\prod_{s=1}^{t}(q^s - q^{-s})} = \begin{Bmatrix}h+c\\t\end{Bmatrix}_q.$$

In particular, for any integer $h \geqslant \max\{-b,-c\}$, the multiplication rule claimed in this theorem specializes to the identity in Theorem 4.3. Since there are infinitely many such values of h, we conclude the proof by applying Lemma 3.2 to the function $g \in \mathbb{Q}(q)[K, K^{-1}]$ defined by

$$g(K,K^{-1}) = \begin{bmatrix}K;c\\t\end{bmatrix}\begin{bmatrix}K;b\\s\end{bmatrix} - \sum_{i\geq 0}\begin{Bmatrix}t-c+b\\i-c\end{Bmatrix}_q\begin{Bmatrix}s-b+c\\i-b\end{Bmatrix}_q\begin{bmatrix}K;i\\t+s\end{bmatrix},$$

treating K as an indeterminant. $\square$

To give the new presentations, we now need one more identity in the Lusztig's integral form, which we state and prove in the proposition below. Recall that $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ and note that the $\mathcal{A}$-algebra $U^0_{\mathcal{A}}(\mathfrak{sl}_2)$ has an automorphism Φ_c for each $c \in \mathbb{Z}$, defined by restricting the automorphism of $\mathbb{Q}(q)[K, K^{-1}]$ which sends $K \mapsto q^c K$ and $K^{-1} \mapsto q^{-c}K^{-1}$. In the following proposition, we set $\begin{bmatrix} K;c \\ t \end{bmatrix} = 0$ if $t < 0$.

Proposition 6.1. *Let $t \geq 1$ and $c \in \mathbb{Z}$. The following relation holds*

$$\begin{bmatrix} K;c+2 \\ t \end{bmatrix} = (q^t + q^{-t}) \begin{bmatrix} K;c+1 \\ t \end{bmatrix} - \begin{bmatrix} K;c \\ t \end{bmatrix} + \begin{bmatrix} K;c \\ t-2 \end{bmatrix}.$$

Proof. First observe that this relation is obtained by applying the automorphism Φ_c to the following relation:

$$\begin{bmatrix} K;2 \\ t \end{bmatrix} = (q^t + q^{-t}) \begin{bmatrix} K;1 \\ t \end{bmatrix} - \begin{bmatrix} K \\ t \end{bmatrix} + \begin{bmatrix} K \\ t-2 \end{bmatrix}. \tag{6.1}$$

We prove this simplified identity by considering two cases, giving full details to indicate an efficient approach.

- Case $t = 1$: by (5.1) we have

$$\begin{aligned}(q + q^{-1}) \begin{bmatrix} K;1 \\ 1 \end{bmatrix} - \begin{bmatrix} K;0 \\ 1 \end{bmatrix} &= (q+q^{-1}) \frac{qK - q^{-1}K^{-1}}{q - q^{-1}} - \frac{K - K^{-1}}{q - q^{-1}} \\ &= \frac{q^2K - q^{-2}K^{-1}}{q - q^{-1}} \\ &= \begin{bmatrix} K;2 \\ 1 \end{bmatrix}.\end{aligned}$$

- Case $t \geq 2$: Multiplying (6.1) by $\{t\}_q!/[K;0] \cdots [K; 2-t+1]$ the left-hand side becomes $[K;2][K;1]$ and the right-hand side becomes $(q^t + q^{-t})[K;1][K;2-t] - [K;2-t][K;1-t] + \{t\}_q\{t-1\}_q$, so it suffices to show that

$$[K;2][K;1] - (q^t + q^{-t})[K;1][K;2-t] + [K;2-t][K;1-t]$$

equals $\{t\}_q\{t-1\}_q$. Multiplying further by $(q - q^{-1})^2$, we obtain

$$\begin{aligned}&(q^2K - q^{-2}K)(qK - q^{-1}K^{-1}) \\ &\qquad - (q^t + q^{-t})(qK - q^{-1}K^{-1})(q^{2-t}K - q^{-2+t}K^{-1}) \\ &\qquad + (q^{2-t}K - q^{t-2}K^{-1})(q^{1-t}K - q^{-1+t}K^{-1})\end{aligned} \tag{6.2}$$

which we must show equals

$$(q-q^{-1})^2\{t\}_q\{t-1\}_q = (q^t - q^{-t})(q^{t-1} - q^{1-t}) = q^{2t-1} + q^{1-2t} - q - q^{-1}.$$

The powers of K that may appear in the expansion of (6.2) are K^2, K^0 and K^{-2}. The coefficient of K^2 is $q^3 - (q^t + q^{-t})q^{3-t} + q^{3-2t} = 0$.

Similarly, the coefficient of K^{-2} is $q^{-3}-(q^t+q^{-t})q^{-3+t}+q^{-3+2t}=0$. Finally, the coefficient of K^0 is

$$\begin{aligned}&-q^{2-1}-q^{-2+1}+(q^t+q^{-t})(q^{1+(-2+t)}+q^{-1+(2-t)})\\&\qquad -q^{(2-t)+(-1+t)}-q^{(t-2)+(1-t)}=q^{2t-1}+q^{1-2t}-q-q^{-1}\end{aligned}$$

as desired. □

Theorem 6.2. *The Cartan subalgebra $\mathcal{U}^0_{\mathcal{A}}(\mathfrak{sl}_2)$ has a presentation given by the generators $\left[\begin{smallmatrix}K;c\\t\end{smallmatrix}\right]$ for $c\in\mathbb{Z}$, $t\in\mathbb{N}_0$, the multiplication relation in Theorem 1.2, and the relation in Proposition 6.1.*

Proof. We need to show that the product of any two generators $\left[\begin{smallmatrix}K;c\\t\end{smallmatrix}\right]\left[\begin{smallmatrix}K;d\\s\end{smallmatrix}\right]$ can be written as an $\mathcal{A}$-linear combination of Lusztig's elements from the relation in Theorem 1.2, and the relation in Proposition 6.1. Notice that by Theorem 1.2 this is true if $t-c+b\geq 0$ and $s-b+c\geq 0$. Moreover $\left[\begin{smallmatrix}K;c\\0\end{smallmatrix}\right]=1$. Therefore, it suffices to show that one can express any $\left[\begin{smallmatrix}K;c\\t\end{smallmatrix}\right]$ for $c\in\mathbb{Z}$ and $t\in\mathbb{N}$ as a linear combination of elements in the $\mathcal{A}$-basis $\mathcal{B}$ in Proposition 5.3(1).

We do this by induction on $t\in\mathbb{N}$. We show that the base case $t=1$ follows by a second induction on c, where the base cases $c=0,1$ hold by definition. If $c\geq 2$, assume that the result is true for $c'<c$. Then we can use Proposition 6.1 to write $\left[\begin{smallmatrix}K;c\\1\end{smallmatrix}\right]$ as an $\mathcal{A}$-linear combination of $\left[\begin{smallmatrix}K;c-1\\1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}K;c-2\\1\end{smallmatrix}\right]$, and so we are done by induction hypothesis on c. If $c<0$, we may assume that the result is true for $c'>c$. We then rearrange the relation in Proposition 6.1 to write

$$\begin{bmatrix}K;c\\1\end{bmatrix}=(q+q^{-1})\begin{bmatrix}K;c+1\\1\end{bmatrix}-\begin{bmatrix}K;c+2\\1\end{bmatrix}$$

so again by induction hypothesis on c, we are done.

Let us now consider the inductive step for $t>1$. By the induction hypothesis on t, the last term $\left[\begin{smallmatrix}K;c\\t-2\end{smallmatrix}\right]$ in Proposition 6.1 can be expressed as an $\mathcal{A}$-linear combination of elements in $\mathcal{B}$. The relation without that term has the same form as for the case $t=1$, and the inductive argument on c for the $t=1$ case carries over verbatim. □

Corollary 6.3. *Lusztig's integral form $\mathcal{U}_{\mathcal{A}}(\mathfrak{sl}_2)$ has a presentation given by the monomials $E^{(n)}\left[\begin{smallmatrix}K;c\\t\end{smallmatrix}\right]F^{(m)}$ for $n,m,t\geq 0$, $c\in\mathbb{Z}$, relations (5.2), (5.3), (5.4), (5.5) and (5.6), together with Proposition 6.1 and the multiplication relation in Theorem 1.2.*

Proof. Relation (5.2) (resp. (5.3)) allows us to rewrite products of divided powers $E^{(n)}E^{(m)}$ (resp. $F^{(n)}F^{(m)}$) as $\mathcal{A}$-multiples of a single divided power. Theorem 6.2 allows us to rewrite products of Lusztig's elements $\left[\begin{smallmatrix}K;c\\t\end{smallmatrix}\right]$ as $\mathcal{A}$-linear combinations of other Lusztig's elements. Finally, relations (5.4), (5.5) and (5.6) allow us to write any product of monomials $E^{(n)}\left[\begin{smallmatrix}K;c\\t\end{smallmatrix}\right]F^{(m)}$ in terms of other such monomials. □

Even though we have stated our results for $\mathcal{U}_{\mathcal{A}}(\mathfrak{sl}_2)$, these bases generalize to arbitrary (finite) rank. Consider Lusztig's form of the quantum group $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$ for an arbitrary Kac–Moody algebra $\mathfrak{g}$ with generalized Cartan matrix of finite rank n, that is, with Cartan datum $(I, \cdot)$ such that $|I| = n$. (The bilinear form $\cdot$ plays no role in the following.) By 1.4.7 and 3.1.13 in [Lus10], its Cartan subalgebra is isomorphic to $U^0_{\mathcal{A}}(\mathfrak{sl}_2)^{\otimes n}$, and we obtain the following corollary.

Corollary 6.4. *Let $\mathfrak{g}$ be a Kac–Moody algebra of finite rank n, and let $K_i^{\pm 1}$ for $i = 1, \ldots, n$ be the generators of the Cartan subalgebra of the quantum group $\mathcal{U}_q(\mathfrak{g})$. Then Lusztig's form for the Cartan subalgebra $U^0_{\mathcal{A}}(\mathfrak{g})$ has an $\mathcal{A}$-basis given by the elements*

$$\begin{bmatrix} K_1; c \\ t_1 \end{bmatrix} \begin{bmatrix} K_2; c_2 \\ t_2 \end{bmatrix} \cdots \begin{bmatrix} K_r; c_r \\ t_r \end{bmatrix}$$

with $c_i \in \{0, \min(1, t_i)\}$. Furthermore, it has a presentation with these generators and the relations in Theorem 6.2 for each set of elements $\left[\begin{smallmatrix} K_i; c \\ t \end{smallmatrix}\right]$, together with the commutativity relations $\left[\begin{smallmatrix} K_i; c \\ t \end{smallmatrix}\right]\left[\begin{smallmatrix} K_j; b \\ s \end{smallmatrix}\right] = \left[\begin{smallmatrix} K_j; b \\ s \end{smallmatrix}\right]\left[\begin{smallmatrix} K_i; c \\ t \end{smallmatrix}\right]$ for $1 \leq i < j \leq n$.

Acknowledgements

Álvaro Gutiérrez was funded by a University of Bristol Research Training Support Grant. Michał Szwej and Mark Wildon gratefuly acknowledge financial support from the Heilbronn Institute for Mathematical Research, Bristol, UK. We would like to thank Christian Krattenthaler for an insightful conversation about hypergeometric series.

References

[AB84] G. E. Andrews and D. M. Bressoud. Identities in combinatorics III: Further aspects of ordered set sorting. *Discrete Math.*, 49(3):223–236, 1984.

[Biz70] M. T. L. Bizley. A generalization of Nanjundiah's identity. *The American Mathematical Monthly*, 77(8):863–865, 1970.

[GKP89] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete mathematics: a foundation for computer science.* Addison-Wesley Longman Publishing Co., Inc., USA, 1989.

[Gou85] I. P. Goulden. A bijective proof of the q-Saalschütz theorem. *Discrete Mathematics*, 57(1):39–44, 1985.

[HK02] J. Hong and S.-J. Kang. *Introduction to Quantum Groups and Crystal Bases.* Advances in the Mathematical Sciences. American Mathematical Society, 2002.

[KC02] V. Kac and P. Cheung. *Quantum calculus.* Universitext. Springer-Verlag, New York, 2002.

[Lus88] G. Lusztig. Quantum deformations of certain simple modules over enveloping algebras. *Adv. in Math.*, 70(2):237–249, 1988.

[Lus90] G. Lusztig. Finite-dimensional Hopf algebras arising from quantized universal enveloping algebra. *J. Amer. Math. Soc.*, 3(1):257–296, 1990.

[Lus10] G. Lusztig. Introduction to quantum groups. Modern Birkhäuser Classics, New York, 2010.

[Nan58] T. S. Nanjundiah. Remark on a note of P. Turán. *The American Mathematical Monthly*, 65(5):354–354, 1958.

[Sta70] R. P. Stanley. *Ordered Structures and Partitions*. PhD thesis, Harvard University, 1970.

[Sta12] R. P. Stanley. *Enumerative Combinatorics: Volume I*, 2nd edition. Cambridge Studies in Advanced Mathematics 49, Cambridge University Press, 2012.

[SY17] M. J. Schlosser and M. Yoo. Basic hypergeometric summations from rook theory. In G. E. Andrews and F. Garvan, editors, *Analytic Number Theory, Modular Forms and q-Hypergeometric Series*, pages 677–692, Cham, 2017. Springer International Publishing.

[Szé85] L. A. Székely. Common origin of cubic binomial identities; A generalization of Surányi's proof on Le Jen Shoo's formula. *J. Comb. Theory Ser. A*, 40(1):171–174, 1985.

[Tak73] L. Takács. On an identity of Shih-Chieh Chu. *Acta Sci. Math. (Szeged)*, 1973.

[Yee08] A. J. Yee. Combinatorial proofs of identities in basic hypergeometric series. *European Journal of Combinatorics*, 29(5):1365–1375, 2008.

[Zei87] D. Zeilberger. A q-Foata proof of the q-Saalschütz identity. *Eur. J. Comb.*, 8(4):461–463, 1987.

[Zen89] J. Zeng. Pfaff–Saalschütz revisited. *J. Comb. Theory Ser. A*, 51(1):141–143, 1989.

University of Bristol
Email address, Á. Gutiérrez: a.gutierrezcaceres@bristol.ac.uk
Email address, M. Szwej: michal.szwej@bristol.ac.uk
Email address, M. Wildon: mark.wildon@bristol.ac.uk

Columbia University
Email address, Á. L. Martínez: alm2297@columbia.edu