An introduction to plethysm

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MFO Miniworkshop 2020

Organizers: Christine Bessenrodt, Chris Bowman, Eugenio Giannelli

Outline

§1 Three settings for plethysms

- Polynomial representations of $\mathrm{GL}(E)$; take $E = \mathbb{C}^3$
- Symmetric functions

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  - $\widetilde{(S^\mu)^{\otimes n}} \otimes S^\nu \big\uparrow_{S_m \wr S_n}^{S_{mn}}$ where $\mu \in \mathrm{Par}(m)$, $\nu \in \mathrm{Par}(n)$

Fourth and fifth? Geometry and invariant theory

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Consider $\mathrm{Sym}^2(\mathrm{Sym}^2 E) \to \mathrm{Sym}^4 E$: $(uv)(u'v') \mapsto uvu'v'$.

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- Such functions are in kernel of $\mathrm{Sym}^4(\mathrm{Sym}^2 E) \to \mathrm{Sym}^8 E$, so

$$\mathrm{Sym}^4(\mathrm{Sym}^2 E) \cong \nabla^{(4,4)}E \oplus \nabla^{(6,2)}E \oplus \nabla^{(8)}E.$$

Defining plethysm by plethystic tableaux

We can define $s_\nu \circ s_\mu$ as the formal character of the composition of Schur functors $\nabla^\nu \circ \nabla^\mu$.

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$$\mathrm{Sym}^4 E \oplus \nabla^{(2,2)}(E) \cong \mathrm{Sym}^2(\mathrm{Sym}^2 E) \leftrightarrow s_{(2)} \circ s_{(2)} = s_{(4)} + s_{(2,2)},$$

we define $s_\nu \circ s_\mu$ by evaluating s_ν at the monomials in s_μ.

- $s_\mu(x_1, x_2, \ldots) = \sum_{t \in \mathrm{SSYT}(\mu)} x^t$
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Define a *plethystic semistandard tableau* of shape μ^ν to be a semistandard ν-tableau whose entries are themselves μ-tableaux. Then

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Plethystic tableaux example

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For example, the plethystic semistandard tableaux of shape $(2)^{(3)}$ and weight $(2, 2, 2)$ are

$$\left\{ \begin{array}{c}
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and so $(s_{(3)} \circ s_{(2)})(x_1, x_2, x_3) = \cdots + 5x_1^2 x_2^2 x_3^3 + \cdots$.

Plethysm defined for symmetric functions

The substitution definition tells us that $(f+g)\circ h = f\circ h + g\circ h$. Moreover, $f\circ p_\ell = p_\ell \circ f$ if f is a positive integral combination of monomials.

Definition

The plethystic product $\circ$ on the ring Λ of symmetric functions is the unique product satisfying

- $p_\ell \circ p_m = p_{\ell m}$
- $(f+g)\circ h = f\circ h + g\circ h$
- $p_\ell \circ (f+g) = p_\ell \circ f + p_\ell \circ g$

for all $f, g, h \in \Lambda$

Highly recommended: N. A. Loehr and J. B. Remmel, *A computational and combinatorial exposé of plethystic calculus*.

§2: Maximal constituents of plethysms

Let $\lambda, \lambda^\star \in \mathrm{Par}(r)$. We say λ *dominates* $\lambda^\star$, and write $\lambda \trianglerighteq \lambda^\star$, if

$$\lambda_1 + \cdots + \lambda_j \geq \lambda^\star_1 + \cdots + \lambda^\star_j.$$

for all j. For example

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Quiz: Choose partitions λ and $\lambda^\star$ of r (a large number) uniformly at random. What is the chance that λ and $\lambda^\star$ are comparable?

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$$\lambda_1 + \cdots + \lambda_j \geq \lambda^\star_1 + \cdots + \lambda^\star_j.$$

for all j. For example

- $(4, 2, 2) \trianglerighteq (3, 3, 1, 1)$,
- $(4, 1, 1)$ and $(3, 3)$ are incomparable.

Quiz: Choose partitions λ and $\lambda^\star$ of r (a large number) uniformly at random. What is the chance that λ and $\lambda^\star$ are comparable?
Answer (Pittel): Almost zero.

Theorem (Paget–W 2016)

The maximal partitions λ such that s_λ appears in $s_\nu \circ s_\mu$ are precisely the maximal weights of the plethystic semistandard tableaux of shape μ^ν.

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In deBoeck–Paget–W, *Plethysms of symmetric functions and highest weight representations*, arXiv 1810.03448 (2018) we used highest weight vectors to give a simpler proof.

Haskell software for enumerating PSSYTs

```
*Example> display $ maximalPSkewTableaux 3 ([3,3],[]) ([2,1],[])
[12,3,3]
11 11 11
2  2  2
11 11 11
3  3  3

[11,5,2]
11 11 11
2  2  2
11 11 12
3  3  2

[10,7,1]
11 11 11
2  2  2
11 12 12
3  2  2

[9,9]
11 11 11
2  2  2
12 12 12
2  2  2
```

Shows that $s_{(3,3)} \circ s_{(2,1)}$ has maximals

$$s_{(12,3,3)}, s_{(11,5,2)}, s_{(10,7,1)}, s_{(9,9)}.$$

§3 Relationships between plethysm coefficients

The Schur functions are an orthonormal basis for the inner product $\langle -, - \rangle$ on symmetric functions.

Theorem (deBoeck–Paget–W 2018)

If r is at least the greatest part of μ then

$$\langle s_\nu \circ s_{(r)\sqcup\mu}, s_{(nr)\sqcup\lambda} \rangle = \langle s_\nu \circ s_\mu, s_\lambda \rangle.$$

- Proved by Newell when $\nu = (n)$ or $\nu = (1^n)$ (1951)
- Proved when $\mu = (1^m)$ by Bruns–Conca–Varbaro (2013)

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Theorem (Brion 1993, deBoeck–Paget–W 2018)

If $r \in \mathbb{N}$ then

$$\langle s_\nu \circ s_{\mu+(1^r)}, s_{\lambda+(n^r)} \rangle \geq \langle s_\nu \circ s_\mu, s_\lambda \rangle$$

- Both proofs determine when the multiplicity stabilises
- Our proof also gives a combinatorial upper bound on the stable multiplicity

§4 Foulkes' Conjecture

Let $\pi^{(m^n)}$ be the permutation character of S_{mn} acting on the set $\Omega^{(m^n)}$ of set partitions of $\{1, 2, \ldots, mn\}$ into n sets each of size m.

Conjecture (Foulkes 1950)

If $m \leq n$ then there is an injection $\mathbb{C}\Omega^{(n^m)} \to \mathbb{C}\Omega^{(m^n)}$

Equivalently

- If $m \leq n$ then $\pi^{(m^n)}$ contains $\pi^{(n^m)}$.
- $\mathrm{Sym}^m(\mathrm{Sym}^n E)$ injects into $\mathrm{Sym}^n(\mathrm{Sym}^m E)$
- $s_{(m)} \circ s_{(n)} - s_{(n)} \circ s_{(m)}$ is a non-negative integral linear combination of Schur functions.

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Foulkes for $n = 2$:

- $\mathrm{Sym}^2\mathrm{Sym}^n E \leftrightarrow s_{(2)} \circ s_{(n)} = s_{(2n)} + s_{(2n-2,2)} + \cdots$
- $\mathrm{Sym}^n\mathrm{Sym}^2 E \leftrightarrow s_{(n)} \circ s_{(2)} = \sum_{\lambda \in \mathrm{Par}(n)} s_{2\lambda}$
- Hence FC is true when $m = 2$ and all n
- These are the only multiplicity-free Foulkes characters for $mn \geq 18$ (Saxl, 1980, W 2009, Godsil–Meagher 2010)

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Progress on Foulkes' Conjecture: $m \leq n$

- True for $m = 3$: Thrall (1942), Dent–Siemons (2000, symmetric group)
- True for $m + n \leq 17$: Mueller–Neunhoffer (2005);
- Kimoto–Lee (2019): explicit highest weight vectors for $\mathrm{Sym}^3\mathrm{Sym}^n E$;
- True for $m = 4$: McKay (2008): the obvious map $\mathbb{C}\Omega^{(n^4)} \to \mathbb{C}\Omega^{(4^n)}$, proposed by Howe (1987), is injective;
- The obvious map $\mathbb{C}\Omega^{(5^5)} \to \mathbb{C}\Omega^{(5^5)}$ is not injective: Mueller–Neunhoffer (2005);
- True for $m + n \leq 19$: Evseev–Paget–W (2014);
- True for $m = 5$: Cheung–Ikenmeyer–Mkrtchyan (2015)

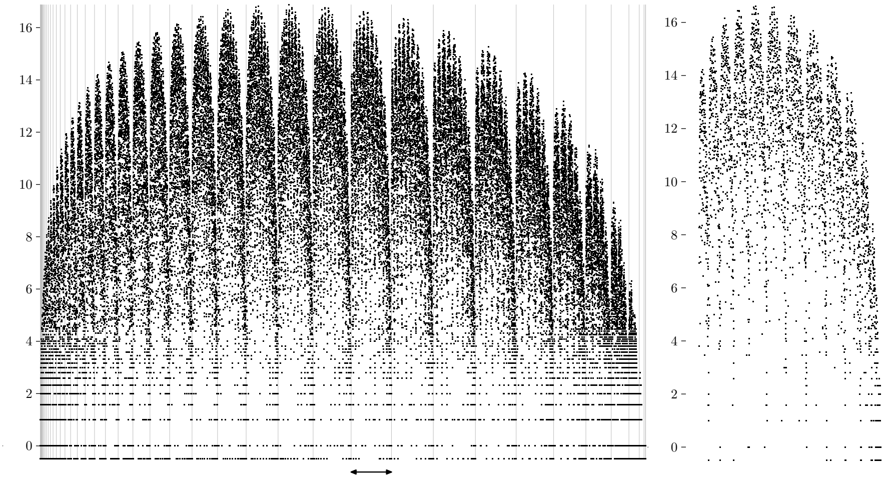
Too easy?

Problem (Stanley Problem 9)

Find a combinatorial rule expressing $s_{(n)} \circ s_{(m)}$ as a non-negative integral linear combination of Schur functions.

Foulkes' Conjecture for $m = 7$, $n = 8$

Logarithms of multiplicities



Foulkes' Conjecture for $m = 7$, $n = 8$

Logarithmic differences in multiplicities: for big dots, smaller multiplicity is 0.